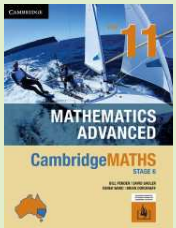
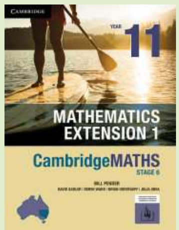
NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS ADVANCED (INCORPORATING EXTENSION 1) YEAR 11 COURSE

Topic summary and exercises:

With references to





Ⓐ **Functions & Graphs**

Name: ....

Initial version by H. Lam, 2018. Last updated June 3, 2024.
Based on the work from the legacy syllabuses by R. Trenwith, 1995–2010, subsequently maintained by H. Lam, 2011-18.
Additional editing by I. Ham and M. Ho, 2019-2020, and A. Sun in late 2020.
Various corrections by students & members of the Mathematics Departments at North Sydney Boys and Normanhurst Boys High Schools.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under ⓒ CC BY 2.0.

Symbols used

- Beware! Heed warning.
- Mathematics Advanced content.
- Mathematics Extension 1 exclusive content.
- Literacy: note new word/phrase.
- Facts/formulae to memorise.
- On the course Reference Sheet.
- ICT usage
- Enrichment content. Broaden your knowledge!

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

MA11-1 uses algebraic and graphical techniques to solve, and where appropriate, compare alternative solutions to problems

MA11-2 uses the concepts of functions and relations to model, analyse and solve practical problems

Syllabus subtopics

MA-F1 Working with Functions

MA-F2 Graphing Techniques

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *Cambridge Year 11 3 Unit* (Pender, Sadler, Shea, & Ward, 1999) will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Contents

Section 1

Functions

Learning Goal(s)

Knowledge
Simplifying surds

Skills
Rewriting surds in lowest base and rationalising

Understanding
The difference between *simplifying* and *rationalising* surds

By the end of this section am I able to:

2.1 Explore the concept of a *function machine*

2.2 Define and use a function and a relation as mappings between sets, and as a rule or a formula that defines one variable quantity in terms of another.

2.3 Identify types of functions and relations on a given domain, using a variety of methods.

2.4 Use function notation, domain and range, independent and dependent variables.

2.5 Piecewise functions

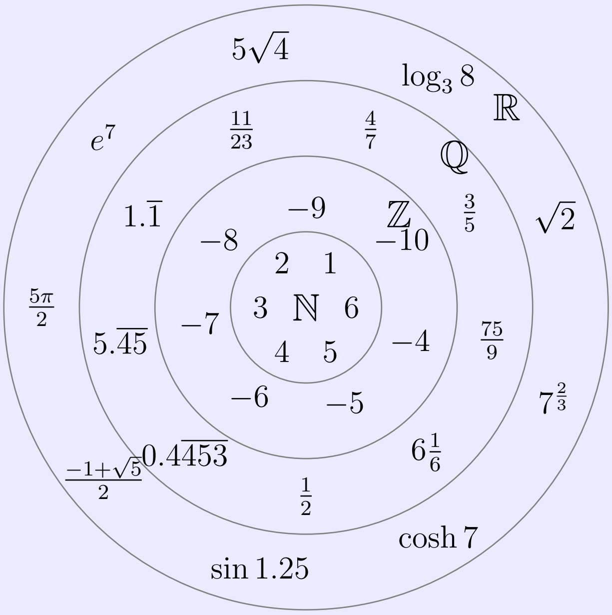
2.6 Understand the concept of the graph of a function

2.7 Define the sum, difference, product and quotient of functions and consider their domains and ranges where possible.

1.1 Interval notation

Definition 1

Types of numbers



- $\mathbb{N}$: natural numbers
- $\mathbb{Z}$: integers
- $\mathbb{Q}$: rational numbers
- $\mathbb{R}$: real numbers

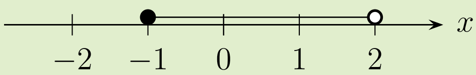
Definition 2

Set notation (read left-to-right)

- $\in$: *belongs to the* ...set... ...of...
- $\cap$: *and* (more formally: ...intersection...)
- $\cup$: *or* (more formally: ...union...)
- $A \subset B$: *A is a* ...subset... *of B*
- $A \supseteq B$: *A is a* ...superset... ...or... ...equal... ...to... B.
- $|A|$: the ...magnitude... (or ...size...) of set A. Refers to the number of unique elements in the set.
- $\overline{A}$, A^c or A': the ...complement... of set A. Everything that is ...not... in the set A.
- $\varnothing$: the ...empty... set. If $|A| = 0$, then $A = \varnothing$
- $[a, b]$: all real numbers from a to b inclusive.
- $(a, b]$: all real numbers from a to b, exclusive at a, inclusive at b.

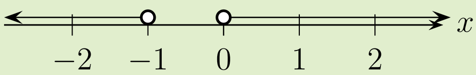
Example 1

Write in interval notation:



Example 2

Write in interval notation:



Exercises

Source These exercises are from Coroneos (1990, Set 2C, p.49-50).

1. Use braces to illustrate with inequality signs the following intervals. Illustrate each interval on the real number line.

(a) $(-3, 0]$ (b) $[0, 1]$ (c) $(0, \infty)$ (d) $[1, 3)$

2. Use parentheses "()" and brackets "[]" to denote

(a) $\{x : 1 \le x < 5\}$ (c) $\{x : x \ge 0\}$

(b) $\{x : x < -1\}$ (d) $\{x : -3 \le x \le 2\}$

3. Which of the operations ($\cup$ or $\cap$) will make the first two intervals, the third?

(a) $(1, 3)$, $(2, 4)$, $(2, 3)$ (b) $[-1, 2)$, $[0, 4)$, $[-1, 4)$

4. Illustrate on the number line:

(a) $(-1, 0]$ (b) $[-1, 2]$ (c) $[-1, 2)$ (d) $(-1, 2)$

5. State whether $A = B$, $A \subset B$ or $B \subset A$ in the following:

(a) $A = \{x : 2 < x < 5\}$, $B = (0, 6)$ (c) $A = (0, 1]$, $B = [0, 1]$

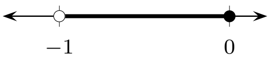
(b) $A = [2, 3]$, $B = (2, 3)$ (d) $A = [0, 1]$, $B = \{x : 0 \le x \le 1\}$

Answers

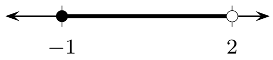
1. (a) $\{x : -3 < x \le 0\}$ (b) $\{x : 0 \le x \le 1\}$ (c) $\{x : x > 0\}$ (d) $\{x : 1 \le x < 3\}$

2. (a) $[1, 5)$ (b) $(-\infty, -1)$ (c) $[0, \infty)$ (d) $[-3, 2]$

3. (a) $\cap$ (b) $\cup$

4. (a)


(b)


(c)


(d)


5. (a) $A \subset B$ (b) $B \subset A$ (c) $A \subset B$ (d) $A = B$

Further exercises

Ⓐ **Ex 10C**
- Q1-14

X1 **Ex 12C**
- Q1-17

1.2 Relations and functions

Definition 3

A **relation** is a ....rule.... that maps ....elements.... from the ....domain.... to the ....range.... .

- The ....domain.... is the set of all possible x values.
- The ....range.... is the set of all possible y values.

Laws/Results

Four types of relations:

- One to one
- One to many
- Many to one
- Many to many

Definition 4

Functions are relations in which each ....input.... (x *-value*) maps to **only one** ....output.... (y *-value*).

Laws/Results

Only ....one.... ....to.... ....one.... and ....many.... ....to.... ....one.... relations are *functions*.

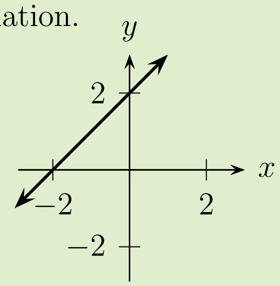
Important note

If a graph is available, use the (informal) *vertical line test.*

Example 3

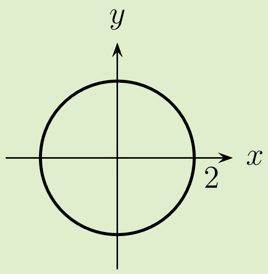
State whether the following are functions or simply relations. Also, classify the type of relation.

(a)



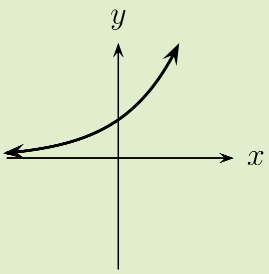
- Function/not a function
- Type: One to one
- Domain: $D = \{x : x \in \mathbb{R}\}$
- Range: $R = \{y : y \in \mathbb{R}\}$

(b)



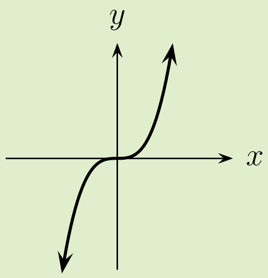
- Function/not a function
- Type: Many to many
- Domain: $D = \{x : -2 \le x \le 2\}$
- Range: $R = \{y : -2 \le y \le 2\}$

(c)



- Function/not a function
- Type: One to one
- Domain: $D = \{x : x \in \mathbb{R}\}$
- Range: $R = \{y : y > 0\}$

(d)

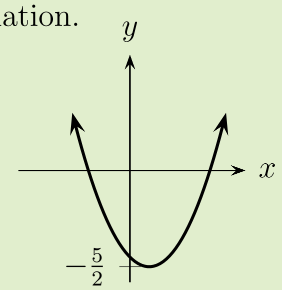


- Function/not a function
- Type: One to one
- Domain: $D = \{x : x \in \mathbb{R}\}$
- Range: $R = \{y : y \in \mathbb{R}\}$

Example 4

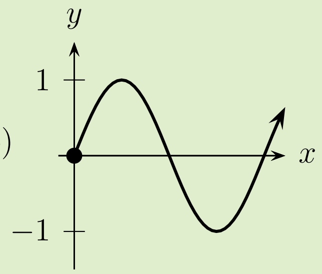
State whether the following are functions or simply relations. Also, classify the type of relation.

(a)



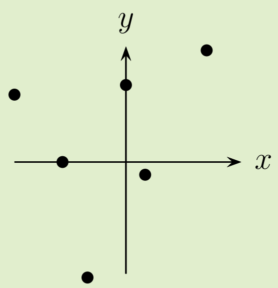
- Function/not a function
- Type: Many to one
- Domain: $D = \{x : x \in \mathbb{R}\}$
- Range: $R = \{y : y \geq -\frac{5}{2}\}$

(b)



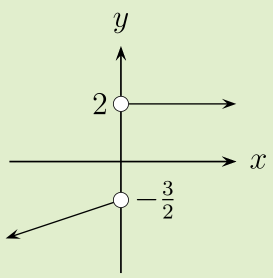
- Function/not a function
- Type: Many to one
- Domain: $D = \{x : x \geq 0\}$
- Range: $R = \{y : -1 \leq y \leq 1\}$

(c)



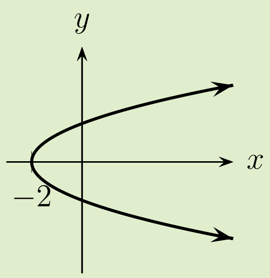
- Function/not a function
- Type: One to one

(d)



- Function/not a function
- Type: Many to one
- Domain: $D = \{x : x \in \mathbb{R}\backslash\{0\}\}$
- Range: $R = \{y : y < -\frac{3}{2} \text{ or } y = 2\}$

(e)



- Function/not a function
- Type: One to many
- Domain: $D = \{x : x \geq -2\}$
- Range: $R = \{y : y \in \mathbb{R}\}$

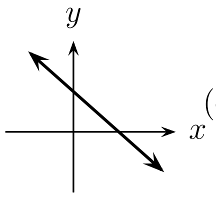
Exercises

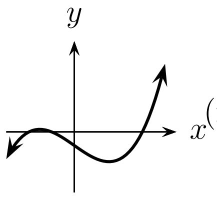
1. Write down the domain and range of each relation:

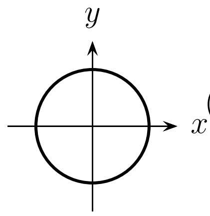
(a) $\{(1,3),(3,0),(5,4)\}$ (b) $\{(1,1),(2,1),(3,1)\}$

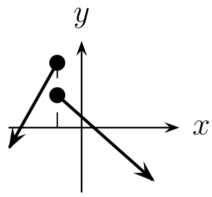
2. State whether each relation is a *function* or *not a function*:

(a) $\{(1,1),(2,2),(3,3)\}$ (b) $\{(1,2),(2,3),(1,4)\}$ (c) $\{(3,2),(4,3),(5,3)\}$

(d) 

(e) 

(f) 

(g) 

(h) $y = 2x$ (i) $x = y^2$ (j) $y = 2$ (k) $x = 1$

Answers

1. (a) $D = \{1,3,5\}$, $R = \{0,3,4\}$. (b) $D = \{1,2,3\}$, $R = \{1\}$ **2. Functions:** (a), (c), (d), (e), (h), (j) **Relations only:** (b), (f), (g), (i), (k)

Further exercises

(A) Ex 3B

- Q1, 2

(X1) Ex 3B

- Q1, 2

1.3 Function notation

Algebraic rule	Name	Placeholder
Dependent variable	Independent variable	

$$f(x) = x^2 - 4x \qquad g(\spadesuit) = \spadesuit^2 - 4\spadesuit$$

Example 5

For the function $f(x) = x^2 + 4x + 1$,

(a) Find $f(1)$

(b) Find $f(-3)$

(c) Find $f(k)$

(d) Find $f(k+1)$

(e) Find $f(3-k)$

(f) Sketch the function

Further exercises

Ⓐ **Ex 3A**

- Q4-5, 7-9, 11-17

(x1) **Ex 3A**

- Q3-4, 6, 8, 9, 10, 12-17, 19

1.3.1 Composition of functions

Learning Goal(s)

Knowledge
What are compositions of functions

Skills
Algebraic manipulation required to simplify the composition of functions

Understanding
The result of composition of two function, in relation to the original functions

By the end of this section am I able to:

2.13 Define and use the composite function $f\big(g(x)\big)$ of functions $f(x)$ and $g(x)$ where appropriate.

Definition 5

If $f(x) = x^2$, then f maps x to x^2, i.e.

$$f : x \mapsto x^2$$

Definition 6

If there are two mappings h and g, then their *composition*

- $h \circ g$: maps g, followed by h
- $g \circ h$: maps h, followed by g

(Draw function chain diagrams and regular function notation to illustrate)

Laws/Results

If f and g are functions and the composite function $f\big(g(x)\big)$ exists:

- The *outer* function is f, the *inner* function is g.
- The ....range.... of the ....inner.... function must be a subset of the ....domain.... of the ....outer.... function.
  - The ....domain.... of the ....inner.... function may need to be restricted
- The ....domain.... of the ....inner.... function (or its restricted version) is the ....domain.... of the composite function.

Example 6

For the functions $f(x) = 7x^2 + 3$ and $g(x) = (2x + 5)^2$, draw a 'function machine' depicting the input/outputs, and also evaluate:

(a) $f\Big(g(x)\Big)$.

(b) $g\Big(f(x)\Big)$.

Example 7

[Ex 13:03] (McSeveney, Conway, & Wilkes, 1986) If $h : x \mapsto x^2$ and $g : x \mapsto x + 3$, then find

(a) $h \circ g(1)$

(b) $g \circ h(-2)$

(c) $g \circ h(a)$

(d) $h \circ g(a)$

Answer: (a) 16 (b) 7 (c) $a^2 + 3$ (d) $(a + 3)^2$

Example 8

[2021 Independent Adv Trial] If $f(x) = \sqrt{x}$ and $g(x) = 4 - x^2$, what is the domain of the function $f\Big(g(x)\Big)$?

(A) $[-2, 2]$

(B) $[0, \infty)$

(C) $(-\infty, 2] \cup [2, \infty)$

(D) $(-\infty, \infty)$

Example 9

[2023 CSSA Adv Trial Q5] Let $f(x) = \sqrt{x}$ and $g(x) = \dfrac{1}{x-1}$.

What is the domain and range for both $f\Big(g(x)\Big)$ and $g\Big(f(x)\Big)$?

(A) **Domain:** $[-1, 1]$ **Range:** $(-\infty, \infty)$

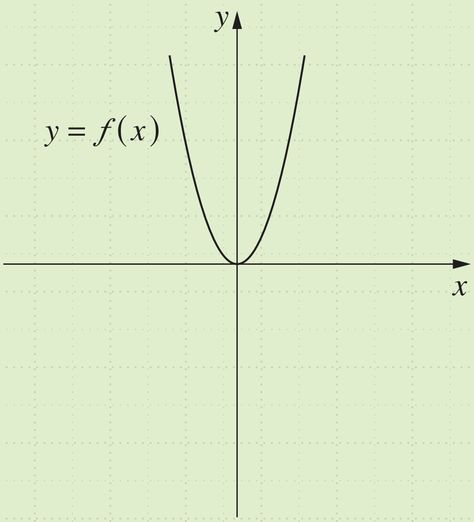
(B) **Domain:** $(0, \infty)$ **Range:** $(-\infty, 0) \cup (-, \infty)$

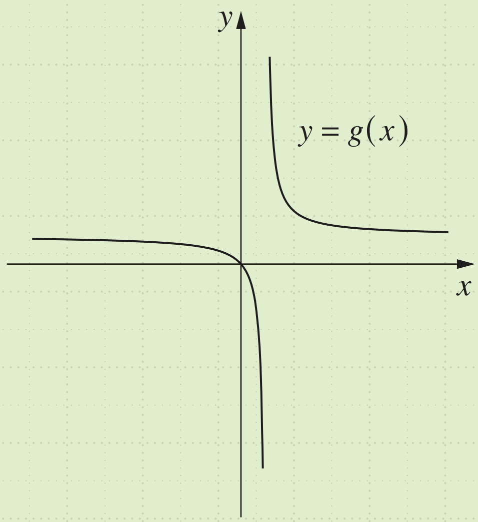
(C) **Domain:** $[0, \infty)$ **Range:** $(0, \infty)$

(D) **Domain:** $(1, \infty)$ **Range:** $(0, \infty)$

Example 10

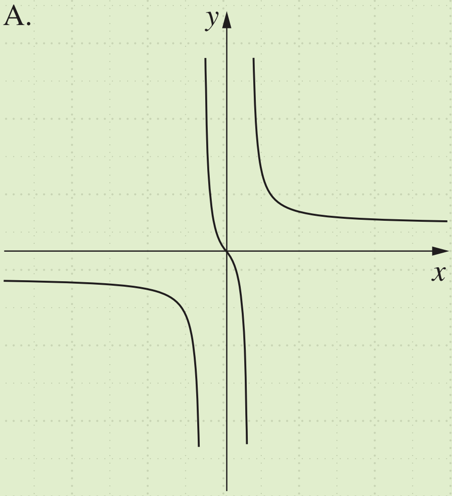
[2021 Adv HSC Q10] The graphs of $y = f(x)$ and $y = g(x)$ are shown.



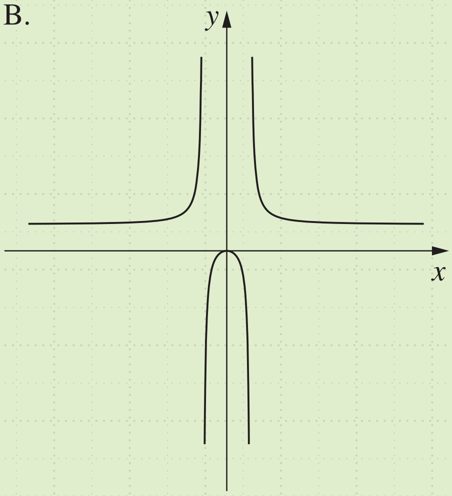


Which of the following best represents $g\Big(f(x)\Big)$?

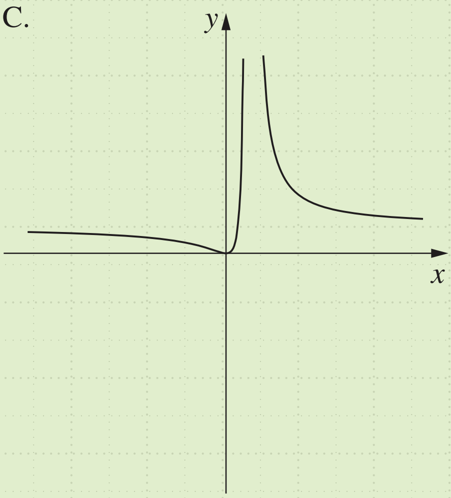
A.



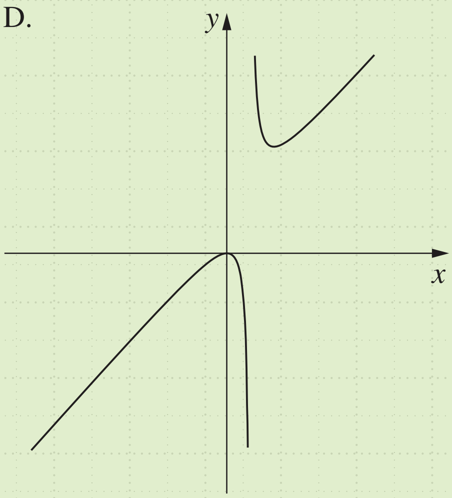
B.



C.



D.



Exercises

1. If $f(x) = 2x^2 - 1$, evaluate:

(a) $f(0)$ (c) $f(-2)$ (e) $f(a)$

(b) $f(3)$ (d) $3f(1) - (f(2))^2$ (f) $f(x+1)$

2. If $g(x) = 2^x + \dfrac{1}{x-1}$,

(a) Evaluate: i. $g(0)$ ii. $g(-4)$ as a fraction.

(b) For what value of x does $g(x)$ not exist?

3. If $f(x) = x^2 + 1$ and $g(x) = 2x - 3$, evaluate and simplify:

(a) $f[g(0)]$ (b) $g[f(0)]$ (c) $f[g(x)]$ (d) $g[f(x)]$

4. Evaluate $h(3)$ if

(a) $h(x) = 5 - x$ (d) $h(x) = 6$

(b) $h(x) = \sqrt{25 - x^2}$ (e) $h(x) = |x|$

(c) $h(x) = 6x$ (f) $h(x) = |2x - 1| - |3 - 5x|$

5. If $f(x) = 3x - 2$, solve

(a) $f(x) = 0$. (b) $f(x) = x + 1$ (c) $f(x) < 1$

6. If $P(x) = x^2 - x - 2$, solve

(a) $P(x) = 0$ (b) $P(x) = 4$

7. If $A(x) = 2x^2 + 7x - 4$ and $B(x) = 2x - 1$, solve $A(x) = B(x)$.

8. If $f(x) = 2^x$, for what value of x does $f(x) = \dfrac{1}{4}$?

9. If $f(x) = x^2 + x$, simplify fully: $\dfrac{f(x+h) - f(x)}{h}$.

10. If $g(x) = x^2 - x - 1$, for what values of x does $g(x) = g(-x)$?

11. If $f(x) = ax + b$, and $f(3) = 2$, $f(4) = 4$, find a and b.

12. If $H : x \mapsto 2x - 1$ and $G : x \mapsto x^2$, find:

(a) $H \circ G(1)$ (b) $G \circ H(1)$ (c) $H \circ G(2)$ (d) $G \circ H(2)$

13. If $m : x \mapsto x - 3$ and $n : x \mapsto x^2$, find:

(a) $m \circ n(x)$ (b) $n \circ m(x)$

(c) the value of x such that $m \circ n(x) = n \circ m(x)$

14. If $h(x) = \frac{1}{x}$, $x \neq 0$ and $g(x) = x - 2$, find a value of x for which $h \circ g(x)$ does not exist.

15. **[2021 CSSA Adv Trial Q14]** Consider the function $f(x) = ax + b$, where a and b are constants. The function satisfies $f(-3) = 9$ and $f(5) = -7$.

(a) Find the values of a and b. **2**

(b) Solve the equation $f\Big(f(x)\Big) = 0$. **1**

The following are sourced from Swale et al. (2019, Ex 3.2), modified to suit NSW syllabuses.

16. If $f(x) = (x-1)(x+3)$ and $g(x) = x^2$, investigate whether the composition function $f\Big(g(x)\Big)$ and $g\Big(f(x)\Big)$ exist. If they do, form the rule for the composite function and state the domain.

17. If $f(x) = 2x - 1$ and $g(x) = \frac{1}{x-2}$, investigate whether the composition function $f\Big(g(x)\Big)$ and $g\Big(f(x)\Big)$ exist. If they do, form the rule for the composite function and state the domain.

18. For the functions $f(x) = x^2 + 1$, $g(x) = \sqrt{x}$ and $h(x) = \frac{1}{x}$, determine whether the following composite functions are defined or undefined. If the composite function exists, identify its domain.

(a) $f \circ g(x)$ (b) $g\Big(f(x)\Big)$ (c) $h\Big(g(x)\Big)$ (d) $h \circ f(x)$

19. For the functions $f(x) = x^2$, $g(x) = \sqrt{x}$ and $h(x) = -\frac{1}{x}$, determine whether the following composite functions are defined or undefined. If the composite function exists, identify its domain.

(a) $f \circ g(x)$ (b) $g\Big(f(x)\Big)$ (c) $h \circ f(x)$ (d) $g\Big(h(x)\Big)$

20. The functions f and g are defined by:

$$f(x) = x^2 + 1 \quad g(x) = \sqrt{x+2}$$

Show that $f\Big(g(x)\Big)$ exists and find the rule for $f\Big(g(x)\Big)$, stating its domain and range.

21. Given the following information:

$$f(x) = \frac{1}{x} \quad D_f = (0, \infty)$$
$$g(x) = \frac{1}{x^2} \quad D_g = R_f$$

(a) Prove that $g\Big(f(x)\Big)$ exists.

(b) Find $g\Big(f(x)\Big)$ and state its domain and range.

(c) Sketch the graph of $y = g\Big(f(x)\Big)$.

22. For the functions $f(x) = \sqrt{x+3}$ and $g(x) = 2x - 5$,

(a) State why $f\Big(g(x)\Big)$ is not defined.

(b) Restrict the domain of g to form a new function, $h(x)$, such that $f\Big(h(x)\Big)$ is defined.

(c) Find $f\Big(h(x)\Big)$

23. For the functions $f(x) = x^2$ and $g(x) = \dfrac{1}{x-4}$

(a) State why $g\Big(f(x)\Big)$ is not defined.

(b) Restrict the domain of f to form a new function, $h(x)$, such that $g\Big(h(x)\Big)$ is defined.

(c) Find $g\Big(h(x)\Big)$

24. If $g(x) = \dfrac{1}{(x-3)^2} - 2$ and $f(x) = \sqrt{x}$,

(a) Prove that $f\Big(g(x)\Big)$ is not defined.

(b) Restrict the domain of g to form a new function, $g_1(x)$, such that $f\Big(g_1(x)\Big)$ exists.

25. For the equations $f(x) = \sqrt{x-4}$ and $g(x) = x^2 - 2$,

(a) Prove that $g\Big(f(x)\Big)$ is defined.

(b) Find the rule for $g\Big(f(x)\Big)$ and state the domain.

(c) Sketch the graph of $y = g\Big(f(x)\Big)$

(d) Prove that $f\Big(g(x)\Big)$ is not defined.

(e) Restrict the domain of g to obtain a function $g_1(x)$ such that $f\Big(g_1(x)\Big)$ exists.

(f) Find $f\Big(g_1(x)\Big)$.

26. Given

$$f(x) = -\sqrt{x} + k \quad D_f = [1, \infty)$$
$$g(x) = x^2 + k \quad D_g = (-\infty, 2]$$

where $k \in \mathbb{R}^+$, find the value(s) of k such that both $f\Big(g(x)\Big)$ and $g\Big(f(x)\Big)$ are defined.

Answers

1. (a) -1 (b) 17 (c) 7 (d) -46 (e) $2a^2 - 1$ (f) $2x^2 + 4x + 1$ **2.** (a) i. 0 ii. $-\frac{11}{80}$ (b) $x = 1$ **3.** (a) 10 (b) -1 (c) $4x^2 - 12x + 10$ (d) $2x^2 - 1$ **4.** (a) 2 (b) 4 (c) 18 (d) 6 (e) 3 (f) -7 **5.** (a) $x = \frac{2}{3}$ (b) $x = \frac{3}{2}$ (c) $x < 1$ **6.** (a) $x = 2, -1$ (b) $x = -2, 3$ **7.** $x = \frac{1}{2}, -3$ **8.** $x = -2$ **9.** $2x + h + 1$ **10.** $x = 0$ **11.** $a = 2$, $b = -4$ **12.** (a) 1 (b) 1 (c) 7 (d) 9 **13.** (a) $x^2 - 3$ (b) $(x-3)^2$ (c) 2 **14.** $x = 2$ **15.** (a) $a = -2$, $b = 3$ (b) $x = \frac{3}{4}$ **16.** $f\Big(g(x)\Big) = (x-1)(x+1)\left(x^2+3\right)$, $D = \mathbb{R}$; $g\Big(f(x)\Big) = (x-1)^2(x+3)^2$, $D = \mathbb{R}$. **17.** $f\Big(g(x)\Big) = \frac{2}{x-2} - 1$, $D = \mathbb{R}\backslash\{2\}$; $g\Big(f(x)\Big)$ does not exist. **18.** (a) $f \circ g(x) = x + 1$, $D = [0, \infty)$ (b) $g\Big(f(x)\Big) = \sqrt{x^2+1}$, $D = \mathbb{R}$ (c) $h\Big(g(x)\Big)$ not defined (d) $h \circ f(x) = \frac{1}{x^2+1}$, $D = \mathbb{R}$ **19.** (a) $f \circ g(x) = x$, $D = [0, \infty)$ (b) $g\Big(f(x)\Big) = |x|$, $D = \mathbb{R}$ (c) $h \circ f(x)$ not defined (d) $g\Big(f(x)\Big)$ not defined **20.** $f\Big(g(x)\Big) = x + 3$, $D = [-2, \infty)$, $R = [0, \infty)$ **21.** (a) $[0, \infty) \subseteq \mathbb{R}\backslash\{0\}$ (b) $g\Big(f(x)\Big) = x^2$, $D = (0, \infty)$, $R = (0, \infty)$ **22.** (a) Range of g is not a subset, or equal set to the domain of f: $\mathbb{R} \not\subseteq [-3, \infty)$ (b) $h(x) = 2x - 5$, $x \in [1, \infty)$ (c) $f\Big(h(x)\Big) = \sqrt{2x-2}$, $x \in [1, \infty)$ **23.** (a) Range of f is not a subset, or equal set to the domain of g: $[0, \infty) \not\subseteq \mathbb{R}\backslash\{4\}$ (b) $h(x) = x^2$, $x \in \mathbb{R}\backslash\{-2, 2\}$ (c) $g\Big(h(x)\Big) = \frac{1}{x^2-4}$, $x \in \mathbb{R}\backslash\{-2, 2\}$ **24.** (a) Range of g is not a subset, or equal set to the domain of f: $(-2, \infty) \not\subseteq [0, \infty)$ (b) $g_(x) = \frac{1}{(x-3)^2} - 2$, $x \in \left(-\infty, -\frac{1}{\sqrt{2}} + 3\right] \cup \left[\frac{1}{\sqrt{2}} + 3, \infty\right)$ (c) $g\Big(h(x)\Big) = \frac{1}{x^2-4}$, $x \in \mathbb{R}\backslash\{-2, 2\}$ **25.** (a) $[0, \infty) \subseteq \mathbb{R}$. Hence, $g\Big(f(x)\Big)$ is defined. (b) $g\Big(f(x)\Big) = x - 6$, $x \in [4, \infty)$. (c) Sketch (d) $[-2, \infty) \not\subseteq [4, \infty)$ (e) $g_1(x) = x^2 - 2$, $x \in \left(-\infty, -\sqrt{6}\right] \cup \left[\sqrt{6}, \infty\right)$ (f) $f\left(g_1(x)\right) = \sqrt{x^2 - 6}$, $x \in \left(-\infty, -\sqrt{6}\right] \cup \left[\sqrt{6}, \infty\right)$. **26.** $k \in [1, 3]$

Further exercises

Ⓐ **Ex 4E**

- Q1-15

ⓧ1 **Ex 4E**

- Q1-17

1.4 Basic functions/relations and their natural domain/range

Definition 7

- **Domain** The set of elements that a relation maps ....from.... .
  - The ....natural.... **domain** of a function is the ....maximum.... set of values for which the function is defined.
- **Range** The set of elements that a relation can map ..to.. .

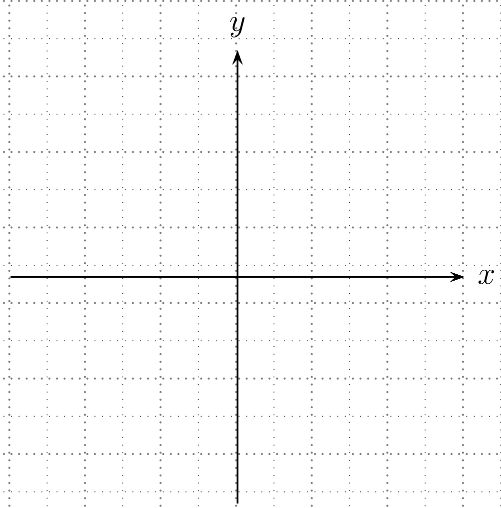
Important note

What is the difference between the domain and natural domain?

Important note

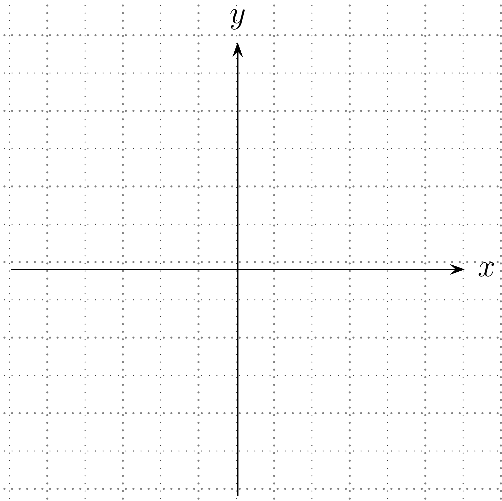
The following curves *must* be known. Sketch the following graphs in their most basic form, and state their natural domain & range using interval notation:

- Straight line $y = x$ **Range:** ....



Natural domain: ....

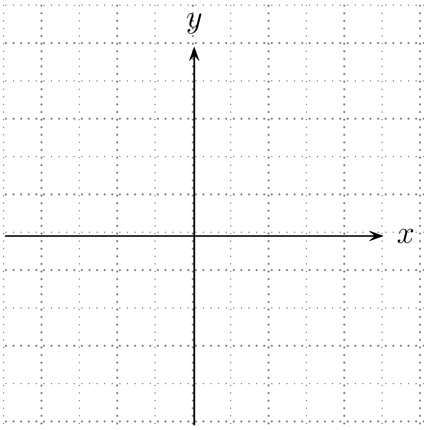
- Parabola $y = x^2$



Natural domain: ....

Range: ....

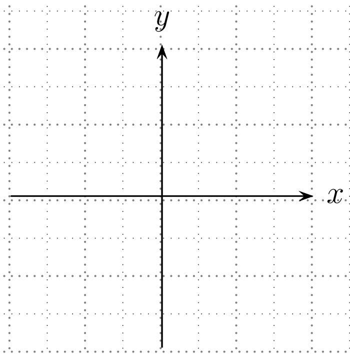
- Basic cubic $y = x^3$



Natural domain: ....

Range: ....

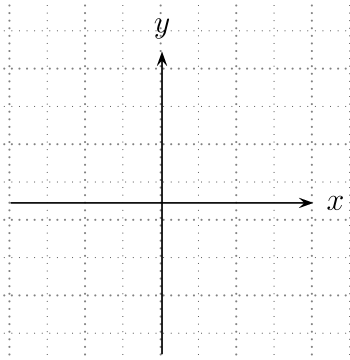
- Hyperbola $y = \frac{1}{x}$



Natural domain: ....

Range: ....

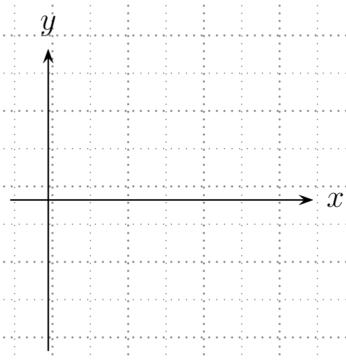
- Circle $x^2 + y^2 = r^2$



Natural domain: ....

Range: ....

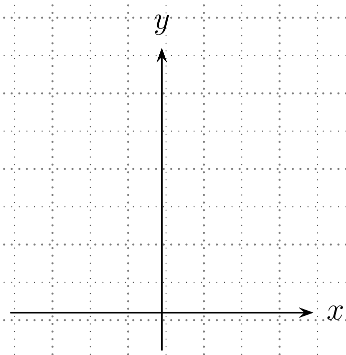
- Logarithmic $y = \log x$



Natural domain: ....

Range: ....

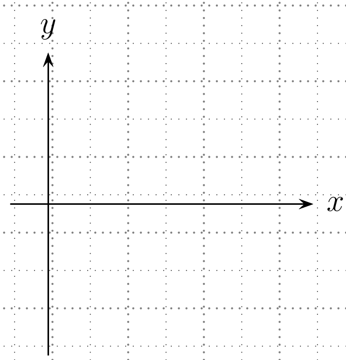
- Exponential $y = a^x$



Natural domain: ....

Range: ....

- Trigonometric $y = \sin x$



Natural domain: ....

Range: ....

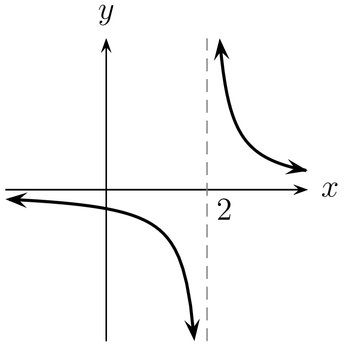
Exercises

1. State the values of x which are *excluded* from the domain of the following:

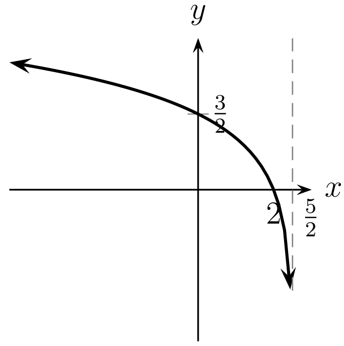
(a) $y = \dfrac{1}{x}$ (b) $y = \sqrt{x}$ (c) $y = \sqrt{9 - x^2}$

2. State the exclusions from the domain shown by the following graphs.

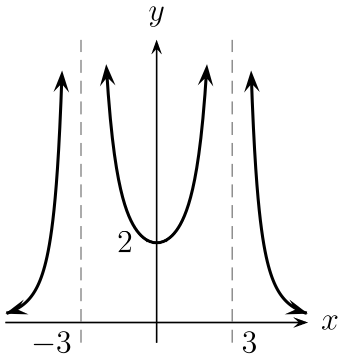
(a)



(c)



(b)



3. State the natural domain of each of the following:

(a) $y = \sqrt{x - 6}$

(b) $y = \dfrac{1}{x+2}$

(c) $y = \sqrt{3x + 1}$

(d) $y = \dfrac{2}{x^2 - 4}$

(e) $y = \sqrt{4 - x^2}$

(f) $y = \dfrac{1}{x+2} + \dfrac{1}{x-3}$

(g) $y = \sqrt{x - 6} + \sqrt{x + 3}$

(h) $y = \dfrac{3}{\sqrt{x - 4}}$

(i) $y = \dfrac{5}{\sqrt{16 - x^2}}$

(j) $y = \dfrac{x}{(x+9)(2x-3)}$

(k) $y = \sqrt{x^2 - 9}$

(l) $y = \dfrac{3}{\sqrt{x^2 - 1}}$

4. For the following functions and relations:

i. Draw neat sketches of the following functions and relations, showing all important features of the graphs

ii. State the natural domain and range using *interval notation*.

(a) $x = 3$

(b) $y = 2x - 4$

(c) $3x + 4y - 12 = 0$

(d) $y = -x^2$

(e) $y = 2x^2 + 1$

(f) $y = (x - 2)^2$

(g) $y = (x - 2)(x - 4)$

(h) $y = x^2 - x - 6$

(i) $y = 8 + 2x - x^2$

(j) $y = (x + 3)^2 + 1$

(k) $y = 4 - (x - 5)^2$

(l) $y = -x^3$

(m) $y = 3 + 2x^3$

(n) $y = x^8$

(o) $y = -x^9$

(p) $x^2 + y^2 = 16$

(q) $x^2 + (y - 3)^2 = 4$

(r) $(x - 1)^2 + (y + 2)^2 = 1$

(s) $x^2 + y^2 + 8x - 6y - 11 = 0$

(t) $x^2 + y^2 - 2x = 0$

(u) $9x^2 + 9y^2 + 9x + 6y + 1 = 0$

(v) $y = 3^x$

(w) $y = -4^{-x}$

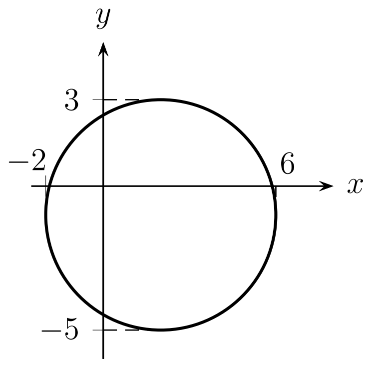
(x) $y = 1 - 2^{-x}$

(y) $y = -\dfrac{3}{x}$

(z) $xy = 8$

5. A standard hyperbola has the x and y axes as its asymptotes, and passes through the point $(3, -2)$. Find the equation.

6. Find the equation of this circle in general form. The numbers on the axes are the extremities of the circle, not the intercepts with the axes.



Further exercises

Ⓐ **Ex 3B**

- Q3, 5, 6, 8, 14

Ⓐ **Ex 3D**

- Q6-9

Ⓐ **Ex 3E**

- Q5

Ⓐ **Ex 3F**

- Q6

(X1) **Ex 3B**

- Q3-5, 8, 13

(X1) **Ex 3D**

- Q7-9

(X1) **Ex 3E**

- Q9

(X1) **Ex 3F**

- Q3

Answers

1. (a) $x = 0$

(b) $x < 0$

(c) $x < -3$ or $x > 3$

2. (a) $x = 2$

(b) $x = \pm 3$

(c) $x \geq \frac{5}{2}$

3. (a) $D = \{x : x \in [6, \infty)\}$

(b) $D = \{x : x \in \mathbb{R} \backslash \{-2\}\}$

(c) $D = \left\{x : x \in \left[-\frac{1}{3}, \infty\right)\right\}$

(d) $D = \{x : x \in \mathbb{R} \backslash \{-2, 2\}\}$

(e) $D = \{x : x \in [-2, 2]\}$

(f) $D = \{x : x \in \mathbb{R} \backslash \{-2, 3\}\}$

(g) $D = \{x : x \in [6, \infty)\}$

(h) $D = \{x : x \in (4, \infty)\}$

(i) $D = \{x : x \in (-4, 4)\}$

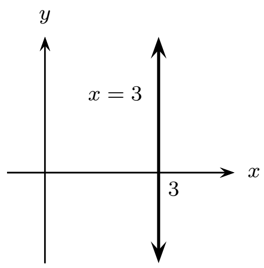
(j) $D = \left\{x : x \in \mathbb{R} \backslash \left\{-9, \frac{3}{2}\right\}\right\}$

(k) $D = \{x : x \in]-3, 3[\}$

(l) $D = \{x : x \in)-1, 1(\}$

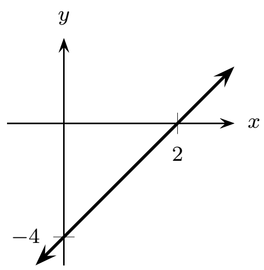
4. (a) $x = 3$

$$D = \{x : x = 3\}$$
$$R = \{y : y \in \mathbb{R}\}$$



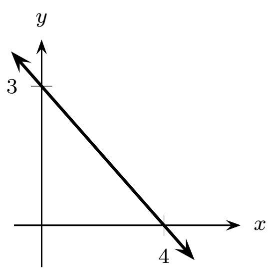
(b) $y = 2x - 4$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in \mathbb{R}\}$$



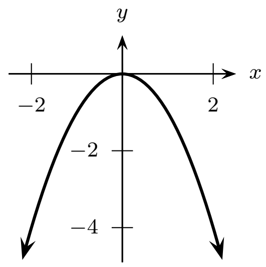
(c) $3x + 4y - 12 = 0$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in \mathbb{R}\}$$



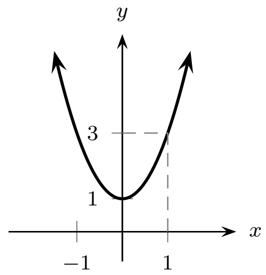
(d) $y = -x^2$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in (-\infty, 0]\}$$



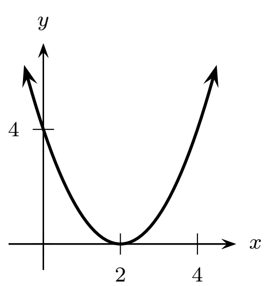
(e) $y = 2x^2 + 1$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in [1, \infty)\}$$



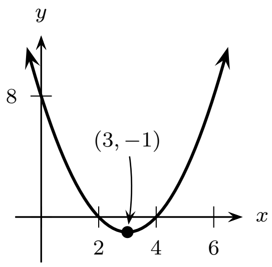
(f) $y = (x - 2)^2$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in [0, \infty)\}$$



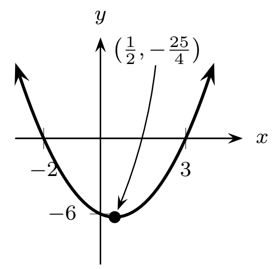
(g) $y = (x - 2)(x - 4)$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in [-1, \infty)\}$$



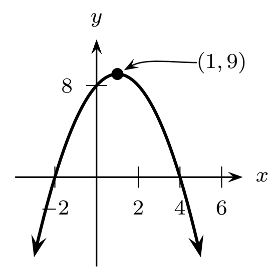
(h) $y = x^2 - x - 6$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \left\{y : y \in \left[-\frac{25}{4}, \infty\right)\right\}$$



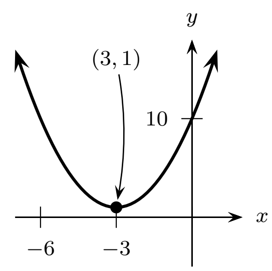
(i) $y = 8 + 2x - x^2$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in (-\infty, 9]\}$$



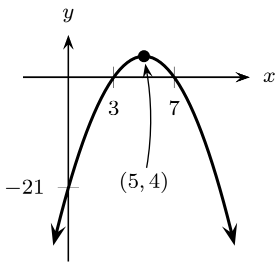
(j) $y = (x + 3)^2 + 1$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in [1, \infty)\}$$



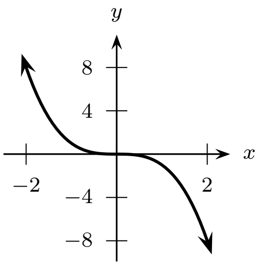
(k) $y = 4 - (x-5)^2$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in (-\infty, 4]\}$$



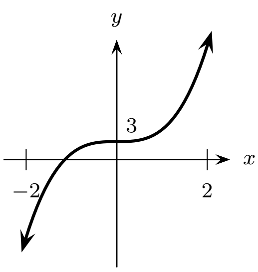
(l) $y = -x^3$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in \mathbb{R}\}$$



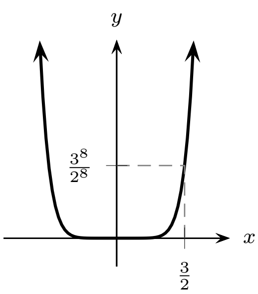
(m) $y = 3 + 2x^3$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in \mathbb{R}\}$$



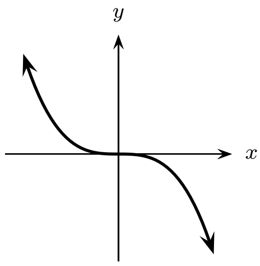
(n) $y = x^8$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in [0, \infty)\}$$



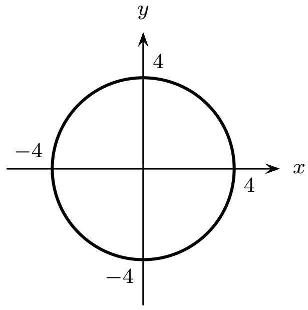
(o) $y = -x^9$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in \mathbb{R}\}$$



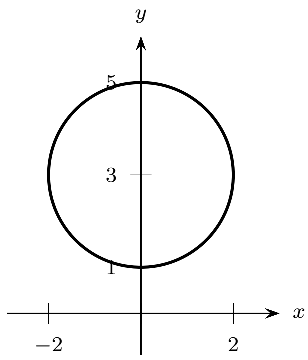
(p) $x^2 + y^2 = 16$

$$D = \{x : x \in [-4, 4]\}$$
$$R = \{y : y \in [-4, 4]\}$$



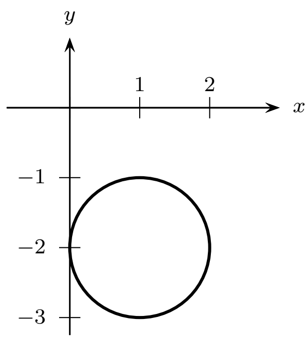
(q) $x^2 + (y-3)^2 = 4$

$$D = \{x : x \in [-2, 2]\}$$
$$R = \{y : y \in [1, 5]\}$$



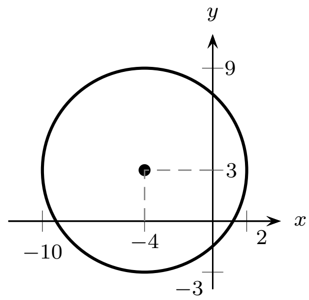
(r) $(x-1)^2 + (y+2)^2 = 1$

$$D = \{x : x \in [0, 2]\}$$
$$R = \{y : y \in [-3, -1]\}$$



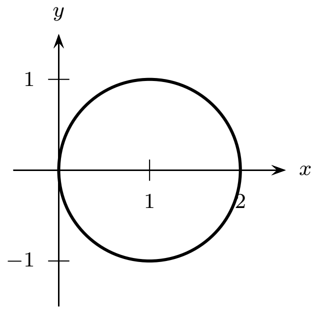
(s) $x^2 + y^2 + 8x - 6y - 11 = 0$

$$D = \{x : x \in [-10, 2]\}$$
$$R = \{y : y \in [-3, 9]\}$$



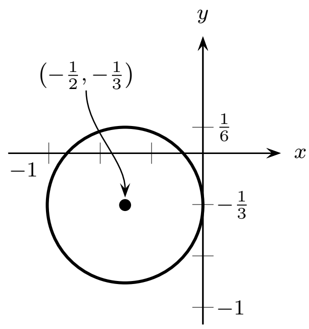
(t) $x^2 + y^2 - 2x = 0$

$$D = \{x : x \in [0, 2]\}$$
$$R = \{y : y \in [-1, 1]\}$$



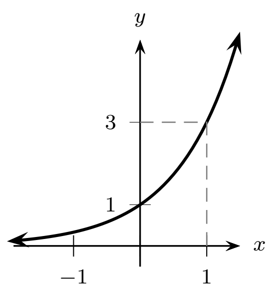
(u) $9x^2 + 9y^2 + 9x + 6y + 1 = 0$

$$D = \{x : x \in [-1, 0]\}$$
$$R = \left\{y : y \in \left[-\frac{5}{6}, \frac{1}{6}\right]\right\}$$



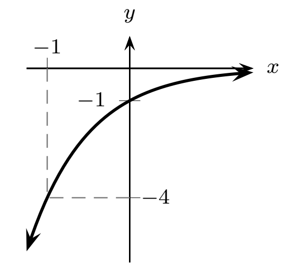
(v) $y = 3^x$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in (0, \infty)\}$$



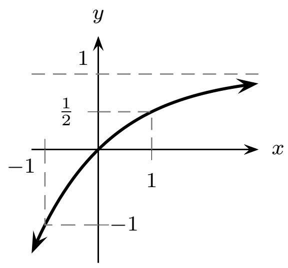
(w) $y = -4^{-x}$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in (-\infty, 0)\}$$



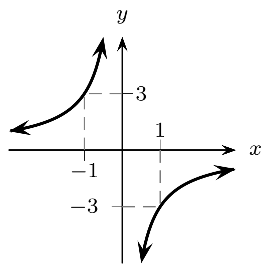
(x) $y = 1 - 2^{-x}$

$$D = \{x : x \in \mathbb{R}\}$$
$$R = \{y : y \in (-\infty, 1)\}$$



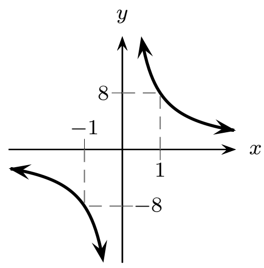
(y) $y = -\dfrac{3}{x}$

$$D = \{x : x \neq 0\}$$
$$R = \{y : y \neq 0\}$$



(z) $xy = 8$

$$D = \{x : x \neq 0\}$$
$$R = \{y : y \neq 0\}$$



5. $xy = -6$

6. $x^2 + y^2 - 4x + 2y - 11 = 0$

1.5 Curve transformations

Learning Goal(s)

Knowledge
How curves are transformed

Skills
Relating the algebra of curve transformation to the graphical representation

Understanding
The differences in algebra when transforming vertically versus horizontally

By the end of this section am I able to:

2.8 Given the graph of $y = f(x)$, sketch $y = -f(x)$ and $y = f(-x)$ and $y = -(f(-x))$ using reflections in the x and y axes.

2.9 Define odd and even functions algebraically and recognise their geometric properties.

Theorem 1

If $a > 0$,

Translations

- $f(x-a)$ shifts right by a units.
- $f(x+a)$ shifts left by a units.
- $f(x)+a$ shifts up by a units.
- $f(x)-a$ shifts down by a units.

Reflections

- $-f(x)$: reflection along the x axis.
- $f(-x)$: reflection along the y axis.

Dilations (stretches)

- $f(ax)$: horizontal dilation
- $af(x)$: vertical dilation

Example 11

Find the centre and radius of $x^2 + y^2 - 6x - 4y + 4 = 0$, and then sketch it.

Example 12

From the graph of $y = \sqrt{x}$, deduce the graph of $y = -\sqrt{-x}$.

Definition 8

Dilate to stretch out and make larger

- Vertical dilation factor of 2: $(x, y) \mapsto (x, 2y)$

  Description: hold x axis constant, then stretch vertically.

- Vertical dilation factor of $\frac{1}{3}$: $(x, y) \mapsto \left(x, \frac{1}{3}y\right)$

  Description: hold x axis constant, then compress vertically.

- Horizontal dilation factor of 2: $(x, y) \mapsto (2x, y)$

  Description: hold y axis constant, then stretch horizontally.

- Horizontal dilation factor of $\frac{1}{5}$: $(x, y) \mapsto \left(\frac{1}{5}x, y\right)$

  Description: hold y axis constant, then compress horizontally.

1.5.1 Horizontal dilations (stretches from y axis)

Steps

To stretch a graph in the horizontal direction by a factor of a from the y axis (*horizontal dilation*), replace x with $\frac{x}{a}$, i.e. new function rule is

$$y = f\left(\frac{x}{a}\right)$$

1.5.2 Vertical dilations (stretches from x axis)

Steps

To stretch a graph in the vertical direction by a factor of a from the x axis (*vertical dilation*), replace y with $\frac{y}{a}$, i.e. new function rule is

$$y = af(x)$$

Example 13

[2022 CSSA Adv Trial Q26] (3 marks) Let $f(x) = x^2 - 2x$.

Sketch the graph of $y = 2f(1 - x) - 6$, showing the location of the vertex.

Exercises

Acknowledgement Portions taken from Evans et al. (2007, Ex 21C)

1. Let $f(x) = 2x + 3$. Sketch the graphs of:

(a) $y = f(x)$
(b) $y = f(x) + 4$
(c) $y = -f(x)$
(d) $y = -f(x) + 2$

2. Let $f(x) = 3x$. Sketch the graphs of:

(a) $y = f(x)$
(b) $y = f(x) + 4$
(c) $y = -f(x)$
(d) $y = -f(-x)$

3. Use transformations to sketch the graphs of each of the following functions and write down its maximal domain and range:

(a) $f(x) = x^2 + 5$
(b) $f(x) = (x - 5)^2$
(c) $f(x) = (x + 4)^2$
(d) $f(x) = x^2 - 3$
(e) $f(x) = 3^{-x}$
(f) $f(x) = 5^x + 1$
(g) $f(x) = 5^x - 4$
(h) $f(x) = 2 + \log_3 x$
(i) $f(x) = \log_3(x - 4)$

4. Sketch the graph of each function and write down its maximal domain and range:

(a) $f(x) = x^2 + 2$
(b) $f(x) = x^2 - 6x + 13$
(c) $f(x) = \sqrt{x}$
(d) $f(x) = \sqrt{2x} + 2$
(e) $f(x) = -\sqrt{x - 2}$
(f) $f(x) = 2 - \sqrt{x + 2}$

5. Let $f(x) = \sqrt{25 - x^2}$. On one set of axes, sketch the graphs of

(a) $y = f(x)$
(b) $y = f(x) + 2$
(c) $y = -f(x)$

6. Let $f(x) = x^3 - 3x^2 + 2x$. On one set of axes, sketch the graphs of

(a) $y = f(x)$
(b) $y = -f(x)$
(c) $y = -2f(x)$

7. Let $f(x) = \dfrac{1}{x}$. On one set of axes, sketch the graphs of

(a) $y = f(x)$
(b) $y = 2f(x)$
(c) $y = -f(x)$
(d) $y = -2f(x)$

Further exercises

Ⓐ **Ex 4A**
- Q3-6, 8, 9, 11, 16, 17

Ⓐ **Ex 4B**
- Q5

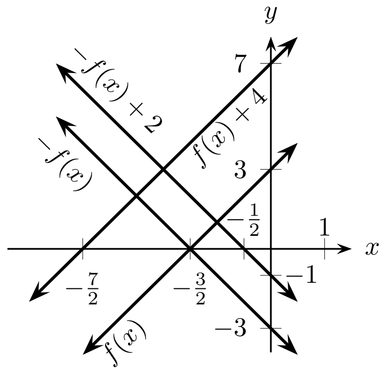
(x1) **Ex 4A**
- Q3, 5, 6, 11-14

(x1) **Ex 4B**
- Q5, 13

Answers

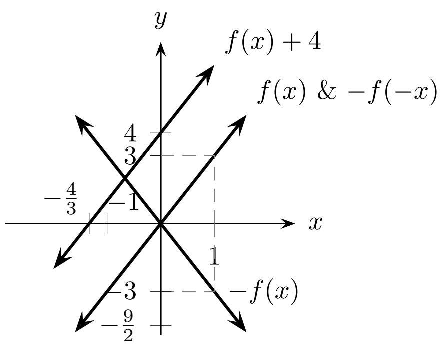
1. (a) $f(x) = 2x + 3$ (c) $-f(x) = -2x - 3$

(b) $f(x) + 4 = 2x + 7$ (d) $-f(x) + 2 = -2x - 1$



2. (a) $y = f(x) = 3x$ (c) $-f(x) = -3x$

(b) $f(x) + 4 = 3x + 4$ (d) $-f(-x) = 3x$



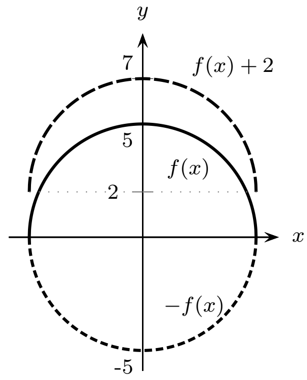
3. No sketch provided here. Instead, the direction & magnitude of shifting is shown with domain/range.

	Orig. fn	**Shift**	**Amt**	$D =$	$R =$
(a)	x^2	$\uparrow$	5	$x \in \mathbb{R}$	$y \geq 5$
(b)	x^2	$\rightarrow$	5	$x \in \mathbb{R}$	$y \geq 0$
(c)	x^2	$\leftarrow$	4	$x \in \mathbb{R}$	$y \geq 0$
(d)	x^2	$\downarrow$	3	$x \in \mathbb{R}$	$y \geq -3$
(e)	3^x	$\leftrightarrow$		$x \in \mathbb{R}$	$y \geq 0$
(f)	5^x	$\uparrow$	1	$x \in \mathbb{R}$	$y > 1$
(g)	5^x	$\downarrow$	4	$x \in \mathbb{R}$	$y > -4$
(h)	$\log_3 x$	$\uparrow$	2	$x > 0$	$y \in \mathbb{R}$
(i)	$\log_3 x$	$\rightarrow$	4	$x > 4$	$y \in \mathbb{R}$

4. No sketch provided here. Instead, the direction & magnitude of shifting + stretching is shown with domain/range.

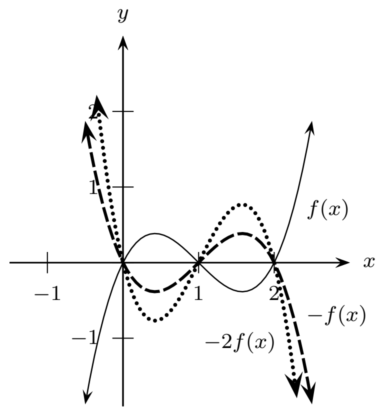
	Orig. fn	**Horiz shift**	**Vert shift**	**Reflect /stretch**	$D =$	$R =$
(a)	x^2	none	$\uparrow 2$	none	$\mathbb{R}$	$y \geq 2$
(b)	x^2	$\rightarrow 3$	$\uparrow 4$	none	$\mathbb{R}$	$y \geq 4$
(c)	$\sqrt{x}$	none	none	none	$x \geq 0$	$y \geq 0$
(d)	$\sqrt{x}$	none	$\uparrow 2$	$\updownarrow 2$	$x \geq 0$	$y \geq 2$
(e)	$\sqrt{x}$	$\rightarrow 2$	none	x axis	$x \geq 2$	$y \leq 0$
(f)	$\sqrt{x}$	$\leftarrow 2$	$\uparrow 2$	x axis	$x \geq -2$	$y \leq 2$

5.

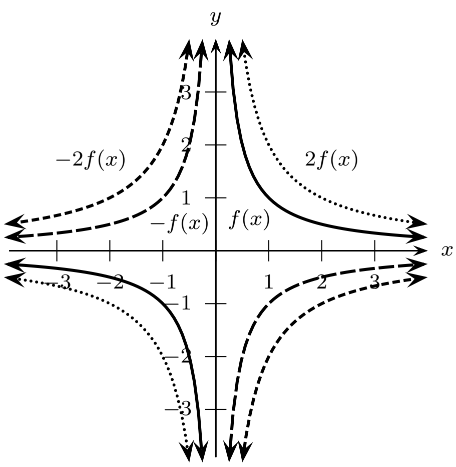


6. $y = x^3 - 3x^2 + 2x$. Factorising,

$$y = x(x-2)(x-1)$$



7. $f(x) = \frac{1}{x}$.



1.6 Odd and even functions

Definition 9

- An odd function has ....point.... ....symmetry........ about the origin. Algebraically,

$$f(-x) = \underline{\ -f(x)\ }$$

- An even function has ....symmetry........ about the y axis. Algebraically,

$$f(x) = \underline{\ f(-x)\ }$$

Important note

Observation:

- Odd polynomials only contain ...odd... powers of x.
- Even polynomials only contain ...even... powers of x.

Example 14

Test these functions to see whether they are odd, even or neither.

(a) $f(x) = x^4 - 3$. (b) $f(x) = x^3$. (c) $f(x) = x^2 - 2x$.

Example 15

[2023 Adv HSC Q9] Let $f(x)$ be any function with domain all real numbers.

Which of the following is an even function, regardless of the choice of $f(x)$?

(A) $2f(x)$ (B) $f\Big(f(x)\Big)$ (C) $\Big(f(-x)\Big)^2$ (D) $f(x)f(-x)$

Exercises

1. State whether the following functions are *even*, *odd* or *neither*:

(a) $y = 2x^2$

(b) $y = 2x^3$

(c) $y = x^4 - x^2 + 1$

(d) $y = x - x^3 + 1$

(e) $y = \dfrac{x}{x^2 - 1}$

(f) $y = (x+1)^2$

(g) $y = 2^x$

(h) $y = 2^{-x}$

(i) $y = 2^x + 2^{-x}$

(j) $y = 2^x - 2^{-x}$

2. (a) Sketch the graph of $y = x^3$, $0 \le x \le 2$. Show the coordinates of the end points.

(b) The above function is part of an even function $f(x)$, defined in the domain $-2 \le x \le 2$. Sketch $y = f(x)$.

(c) Write a piecewise description of the function $f(x)$.

3. (a) Sketch the graphs of $y = \sin x$ and $y = \cos x$ on separate number planes, each in the domain $-180^\circ \le x \le 180^\circ$.

(b) Hence decide if the sine and cosine functions are odd, even or neither.

(c) Remembering that $\sin 45^\circ = \cos 45^\circ = \dfrac{1}{\sqrt{2}}$, write down the values of $\sin(-45^\circ)$ and $\cos(-45^\circ)$.

4. Show that if an odd function is defined for $x = 0$, then its graph must pass through the origin.

Further exercises

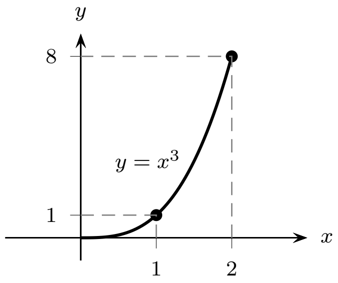
Ⓐ **Ex 4C**
- Q1-7, 9-10, 12-13

(X1) **Ex 4C**
- Q1-7, 9-10, 12(a), 13-15

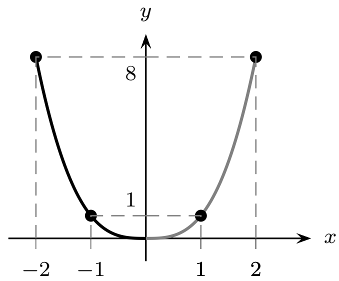
Answers

1. (a) even (b) odd (c) even (d) neither (e) odd (f) neither (g) neither (h) neither (i) even (j) odd

2. (a) $y = x^3$, $0 \le x \le 2$

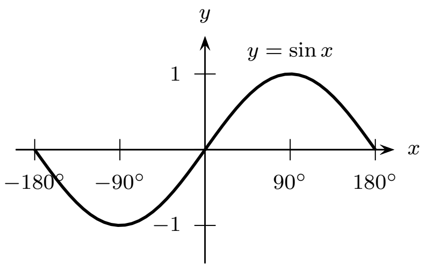


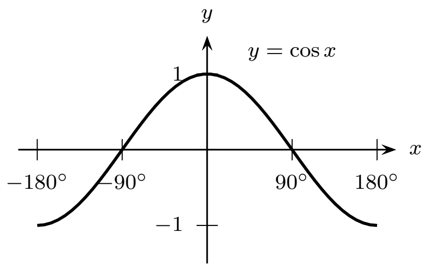
(b) Reflect $y = x^3$, $0 \le x \le 2$ about the y axis:



(c) $f(x) = \begin{cases} -x^3 & -2 \le x < 0 \\ x^3 & 0 \le x \le 2 \end{cases}$

3. (a) $y = \sin x$, $y = \cos x$, $-180° \le x \le 180°$





(b) $y = \sin x$: odd. $y = \cos x$: even

(c) $\sin(-45°) = -\sin 45° = -\frac{1}{\sqrt{2}}$.
$\cos(-45°) = \cos 45° = \frac{1}{\sqrt{2}}$.

Section 2

Various families of functions

Learning Goal(s)

Knowledge
All basic curves

Skills
Sketching these basic curves with transformations

Understanding
Basic applications of these curves

By the end of this section am I able to:

2.14 Model, analyse and solve problems involving linear functions.

2.15 Recognise linear functions

2.16 Sketch linear functions

2.17 Model, analyse and solve problems involving quadratic functions.

2.18 Solve practical problems involving a pair of simultaneous linear and/or quadratic functions algebraically and graphically, with or without the aid of technology; including determining and interpreting the break-even point of a simple business problem

2.19 Recognise cubic functions of the form: $f(x) = kx^3$, $f(x) = k(x-b)^3 + c$, and $f(x) = k(x-a)(x-b)(x-c)$ where a, b, c and k are constants, from their equation and/or graph and identify important features of the graph.

2.20 Define a real polynomial $P(x)$ as the expression $a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n$ where $n = 0, 1, 2, \cdots$ and $a_0, a_1, \cdots, a_n$ are real numbers.

2.21 Identify the coefficients and the degree of a polynomial.

2.22 Identify the shape and features of graphs of polynomial functions of any degree in factored form and sketch their graphs.

2.23 Recognise features of the graphs of $x^2 + y^2 = r^2$ and $(x-a)^2 + (y-b)^2 = r^2$, including their circular shapes, their centres and their radii.

2.24 Recognise that functions of the form $f(x) = \frac{k}{x}$ represent inverse variation, identify the hyperbolic shape of their graphs and identify their asymptotes.

2.25 Exponential functions and asymptotes of the exponential function.

2.26 Rectangular hyperbolas and asymptotes of the hyperbola.

2.27 Early introduction to limit expressions involving $\lim_{x\to 0} f(x)$, $\lim_{x\to\infty} f(x)$, as well as continuity.

Important note

Most of these families of curves have already been covered in the previous sections. Refer to previous textbook exercises for further practice.

2.1 Linear functions

Laws/Results

The **gradient-intercept form** of the equation of a line:

$$y = mx + b$$

where

- m – gradient
- b – y intercept

Laws/Results

The **general form** of the equation of a line:

$$ax + by + c = 0$$

Laws/Results

The **point-gradient formula** to find the equation of a straight line through (x_1, y_1) with a given gradient m:

$$\frac{y - y_1}{x - x_1} = m$$

Derivation of formula:

Important note

No new theory is in this section. However, expect slightly more difficult problems within textbook exercises.

Example 16

[Ex 5D Q6] Given the points $A(1,-2)$ and $B(-3,4)$, find in general form the equation of:

(a) the line AB,

(b) the line through A perpendicular to AB.

Example 17

[Ex 5D Q12]

(a) On a number plane, plot the points $A(4,3)$, $B(0,-3)$ and $C(4,0)$.

(b) Find the equation of BC.

(c) Explain why $OABC$ is a parallelogram.

(d) Find the area of $OABC$ and the length of the diagonal AB.

Further exercises

(A) **Ex 3C**

- All questions

(X1) **Ex 1F**

- Q2-4

(X1) **Ex 3C**

- Q1-13

2.2 Piecewise defined functions, absolute value

Learning Goal(s)

Knowledge
What absolute value functions are

Skills
Transforming between algebraic and graphical representations, solve absolute value equations

Understanding
Absolute value functions are the reflection of the original graph along the horizontal axis.

By the end of this section am I able to:

2.10 Define the absolute value $|x|$ of a real number x as the distance of the number from the origin on a number line without regard to its sign.

2.11 Use and apply the notation $|x|$ for the absolute value of the real number x and the graph of $y = |x|$

2.12 Solve simple absolute value equations of the form $|ax + b| = k$ both algebraically and graphically.

Definition 10

A *piecewise defined function* is one that has multiple definitions depending on the input values.

Example 18

Sketch $f(x) = \begin{cases} x^2 & x < 0 \\ 3x & 0 \le x < 4 \\ 12 & x > 4 \end{cases}$.

Example 19

[**2023 Independent Adv Trial Q3**] A function f is given by the rule

$$f(x) = \begin{cases} (x-2)^2 + 1 & x \leq 2 \\ 4x - 7 & x > 2 \end{cases}$$

What is the value of $f\Big(f(0)\Big)$?

(A) 5 (B) -7 (C) -35 (D) 13

2.2.1 The absolute value function

Definition 11

The absolute value function

$$|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

Important note

- Geometrically, reflect the ....negative.... parts of the graph about the ..x.. axis.
- Also be ready to see the graph as a transformation.

Example 20

Sketch $y = |x - 2|$

Example 21

Sketch $y = |2x - 1|$.

Example 22

Sketch $y = |x^2 - 2x - 8|$.

Example 23

[2021 Independent Adv Trial Q2] What is the number of distinct solutions of the equation $|x^2 - 1| = 1$?

(A) 1 (B) 2 (C) 3 (D) 4

Exercises

1. Sketch each function over the stated domain. State also the range of the function over the specified domain:

(a) $y = 3 - 2x,\ x \geq 1$

(b) $y = x^2,\ 0 \leq x \leq 2$

(c) $xy = 6,\ -2 < x \leq 3,\ x \neq 0.$

(d) $y = (x+2)^2 - 1,\ -3 \leq x \leq 0.$

2. Sketch each of the following piecewise defined functions, showing the coordinates of the endpoints of each interval. State also the range of the function over the specified domain.

(a) $f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ -x+1 & \text{if } x \geq 0 \end{cases}$

(b) $f(x) = \begin{cases} 2x+1 & \text{if } x < 1 \\ 3 & \text{if } x \geq 1 \end{cases}$

(c) $f(x) = \begin{cases} -2-x & \text{if } x \leq 2 \\ 2x-3 & \text{if } x > 2 \end{cases}$

(d) $f(x) = \begin{cases} x^2+1 & \text{if } x < 0 \\ 1-x & \text{if } x \geq 0 \end{cases}$

(e) $f(x) = \begin{cases} -x^2 & \text{if } x < 1 \\ 2^x & \text{if } x > 1 \end{cases}$

(f) $f(x) = \begin{cases} -2x-3 & \text{for } x < -1 \\ -1 & \text{for } -1 \leq x < 1 \\ -\dfrac{1}{x} & \text{for } x \geq 1 \end{cases}$

3. If $f(x) = \begin{cases} 1-x & \text{if } x < 3 \\ x^2+2 & \text{if } x \geq 3 \end{cases}$, evaluate (a) $f(-5)$ (b) $f(3)$ (c) $f(5)$

4. If $f(x) = \begin{cases} x^2-1 & \text{if } x \leq -4 \\ 6 & \text{if } -4 < x < 0 \\ 5x & \text{if } x \geq 0 \end{cases}$, evaluate

(a) $f(-6)$

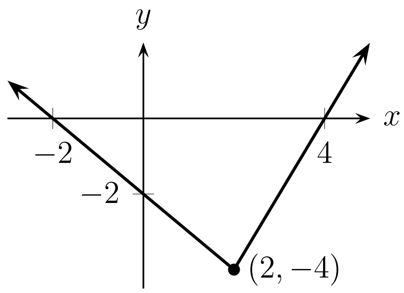
(b) $f(-2)$

(c) $f(-4) + f(2)$

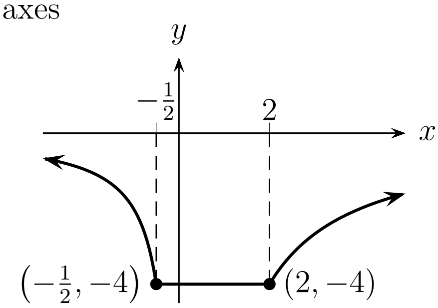
(d) $f(a^2)$

5. Write down the piecewise descriptions for the following functions:

(a)



(b) Both curved sections are hyperbolae whose asymptotes are the x and y axes



6. Sketch the following functions, stating the domain and range in each case:

(a) $y = |x-4|$

(b) $y = -|x+3|$

(c) $y = |x| + 3$

(d) $y = |2x-3|$

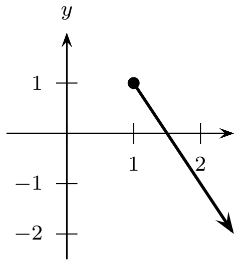
(e) $y = 2|x| - 3$

(f) $y = |3x+7| - 2$

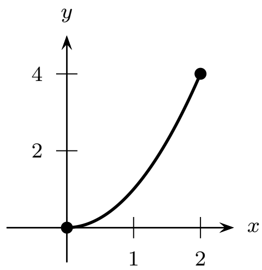
(g) $y = 4 - 3|4-x|$

Answers

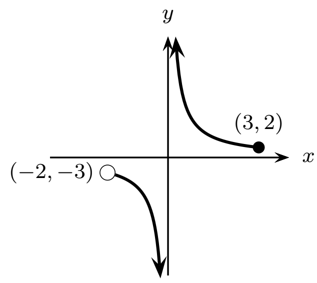
1. (a) $R = \{y : y \le 1\}$.



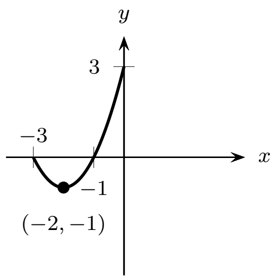
(b) $R = \{y : 0 \le y \le 4\}$.



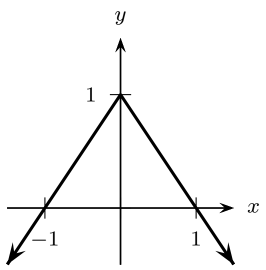
(c) $R = \{y : y < -3, y \ge 2\}$



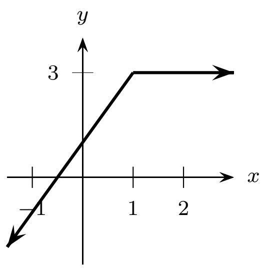
(d) $R = \{y : -1 \le y \le 3\}$



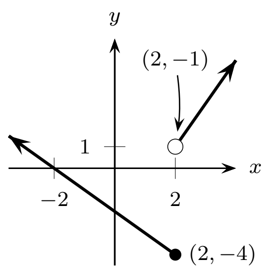
2. (a) $R = \{y : y \le 1\}$



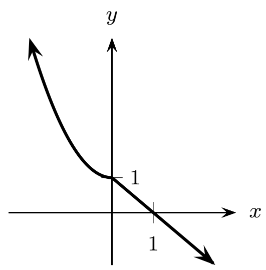
(b) $R = \{y : y \le 3\}$



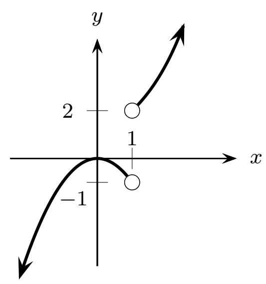
(c) $R = \{y : y \ge -4\}$



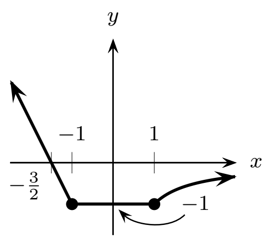
(d) $R = \{y : y \in \mathbb{R}\}$.



(e) $R = \{y : y \le 0\} \cup \{y : y > 2\}$



(f) $R = \{y : y \ge -1\}$



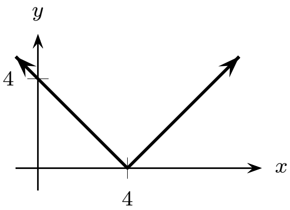
3. (a) 6 (b) 11 (c) 27

4. (a) 35 (b) 6 (c) 25 (d) $5a^2$

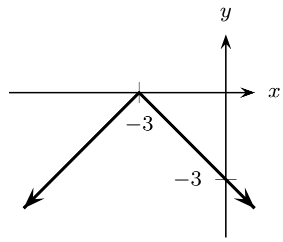
5. (a) $f(x) = \begin{cases} -x-2 & \text{if } x < 2 \\ 2x-8 & \text{if } x \ge 2 \end{cases}$

(b) $f(x) = \begin{cases} \frac{2}{x} & \text{if } x < -\frac{1}{2} \\ -4 & \text{if } -\frac{1}{2} \le x < 2 \\ -\frac{8}{x} & \text{if } x \ge 2 \end{cases}$

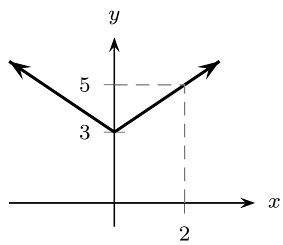
6. (a) $R = \{y : y \geq 0\}$



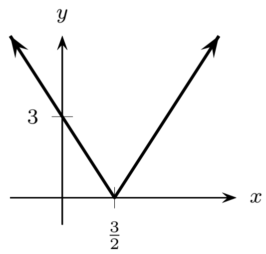
(b) $R = \{y : y \leq 0\}$



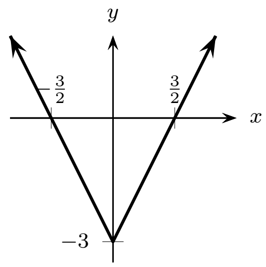
(c) $R = \{y : y \geq 3\}$



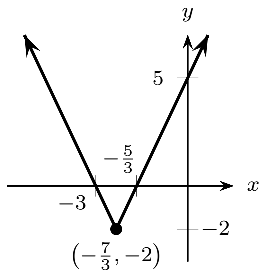
(d) $R = \{y : y \leq 0\}$



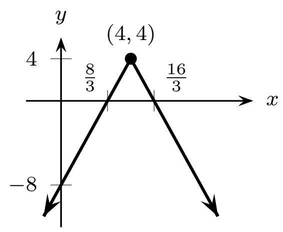
(e) $R = \{y : y \geq -3\}$



(f) $R = \{y : y \geq -2\}$



(g) $R = \{y : y \leq 4\}$



Further exercises

(A) Ex 3B

- Q15

(A) Ex 4D

- Q1, 4, 9-11, 14, 15

(X1) Ex 3A

- Q7 (piecewise defined functions)

(X1) Ex 4D

- Q1, 4, 7, 14, 15, 16, 17, 19

2.2.2 Solving absolute value equations

Important note

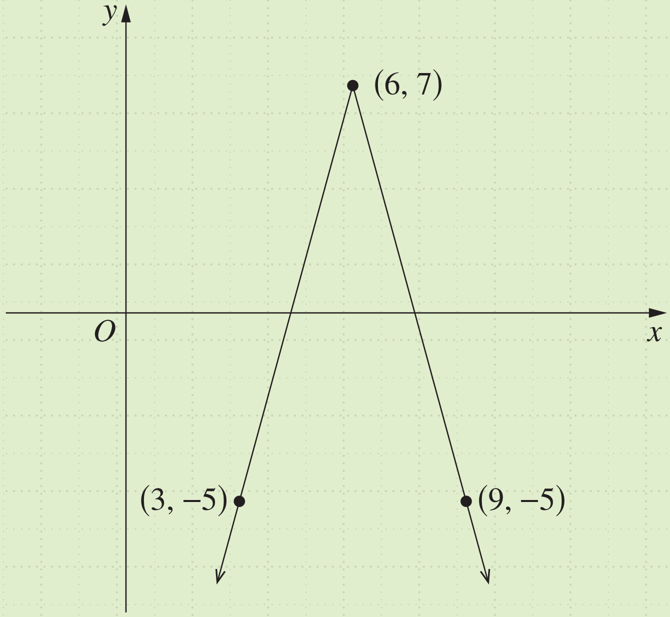
- Draw picture!
- When using purely algebraic methods, test values on equations with absolute value on one side and the other side with algebraic terms without absolute value.

Example 24

Solve $|2x - 1| = 3$. **Answer:** $x = -1, 2$

Example 25

[2023 Adv HSC Q27] The graph of $y = f(x)$, where $f(x) = a|x - b| + c$, passes through the points $(3, -5)$, $(6, 7)$ and $(9, -5)$ as shown in the diagram.



(a) Find the values of a, b and c. **3**

(b) The line $y = mx$ cuts the graph of $y = f(x)$ in two distinct places. **2**

Find all possible values of m.

Exercises

Solve for real values of x:

1. $|x| = 4$

2. $|6x| = 18$

3. $\left|\frac{1}{2}x\right| = 8$

4. $|x - 3| = 0$

5. $|x + 2| = 12$

6. $|x - 6| = 7$

7. $|x + 7| = 4$

8. $|x - 5| = 2$

9. $|2x - 7| = 3$

10. $|3x + 4| = 10$

11. $|1 - 2x| = 5$

12. $|7 - 3x| = 1$

Answers

1. $x = \pm 4$ **2.** $x = \pm 3$ **3.** $x = \pm 16$ **4.** $x = 3$ **5.** $x = 10, -14$ **6.** $x = 13, -1$ **7.** $x = -3, -11$ **8.** $x = 3, 7$ **9.** $x = 2, 5$ **10.** $x = 2, -\frac{14}{3}$ **11.** $x = 3, -2$ **12.** $x = 2, \frac{8}{3}$

2.3 Quadratic functions

2.3.1 Geometry of a parabola

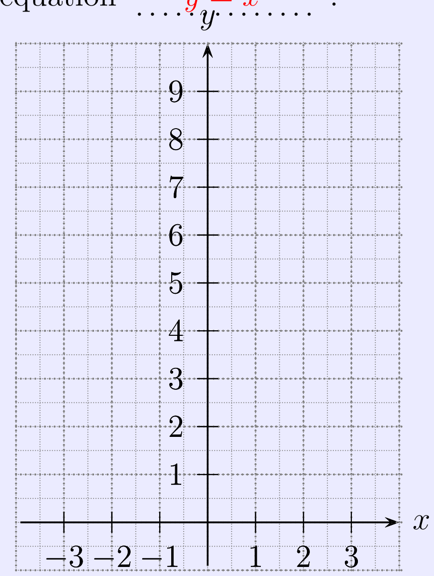
Definition 12

A *parabola* is the geometric representation (graph) of the *quadratic polynomial.*

Basic parabola

Definition 13

The basic parabola has equation $y = x^2$.



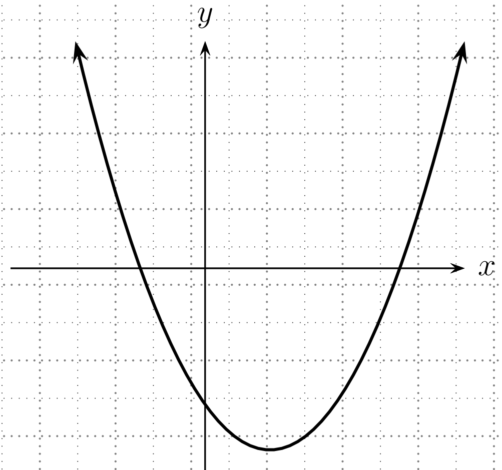
- The **vertex** (minimum value) is at $(0, 0)$
- The **axis of symmetry** is at $x = 0$

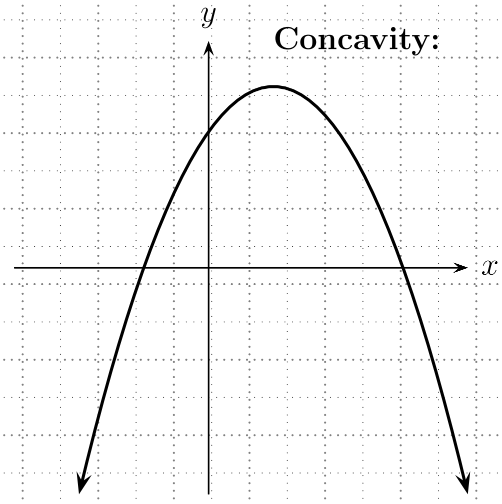
GeoGebra

quadratic.ggb
www.desmos.com

General parabolas

x intercept	**y intercept**	**minimum value (y)**	
maximum value (y)	**turning point**	**vertex**	**axis**





General form and roots

Definition 14

The *general form* of a quadratic polynomial:

$$y = \ldots\ ax^2 + bx + c\ \ldots$$

where c is the y … intercept … of the parabola.

The highest power (degree) of a quadratic: 2

Definition 15

The *roots* or *zeros* of the quadratic polynomial are the x intercepts, if any.

Graph representative of the general form:

2.3.2 Factorisation and the graph

Definition 16

The *factored form* of quadratic polynomial:

$$y = a(x-\alpha)(x-\beta)$$

where

- α and β are the *roots* of the quadratic polynomial
- a is the vertical stretch factor.

Graph representative of the factored form:

Example 26

Find the roots of $y = x^2 + 5x + 6$.

Important note

Factorisation techniques from Stage 4-5 (Years 8-10) are required!

Example 27

Sketch the following graphs:

(a) $y = x^2 - 6x + 8$

(b) $y = -2x^2 + 9x - 7$

Example 28

Solve $x^2 - 6x + 5 \geq 0$.

Important note

⚠ Draw picture!

Example 29

[Ex 8A Q7] Write down the general form of a monic quadratic for which one of the zeroes of $x = 1$. Then find the equation of such a quadratic in which

(a) the curve passes through the origin,

(b) there are no other zeroes,

(c) the axis of symmetry is $x = -7$,

(d) the curve passes through $(3, 9)$.

Laws/Results

The vertical stretch factor needs to be considered when two roots are given and the parabola's equation needs to be formed.

Example 30

Write down the equation of a parabola with zeroes -2 and 4, as well as

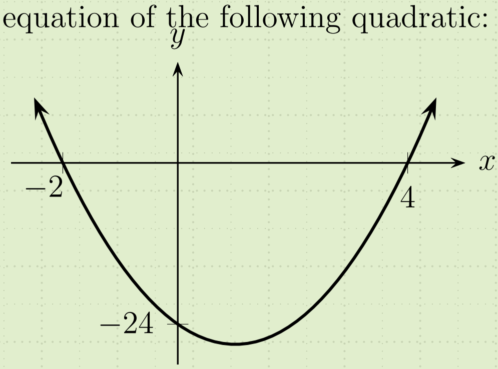
(a) y intercept of 6

(b) vertex at $(1, 21)$

Answer: (a) $a = -\frac{3}{4}$ (b) $a = -\frac{7}{3}$

Example 31

[Ex 8A Q12] Find the equation of the following quadratic:



Further exercises

(A) Ex 3D

- All questions

(X1) Ex 3D

- Q5-16
- Q17, 18

2.3.3 Completing the square and the graph

Definition 17

The *vertex form* of quadratic polynomial:

$$y = a(x-h)^2 + k$$

where

- (h, k) is the vertex of the parabola
- a is the vertical stretch factor.

Graph representative of the vertex form:

Example 32

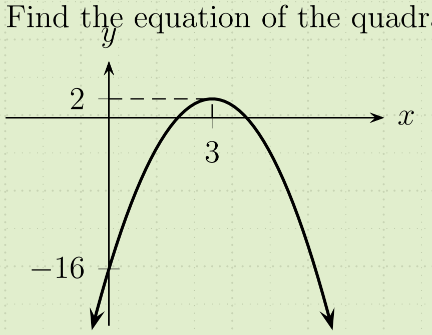
Sketch the following parabolas:

(a) $y = x^2 + x + 1$

(b) $y = -3x^2 - 4x + 2$

Example 33

Find the equation of the quadratic function:



Further exercises

Ex 3E (A)
- All questions

Ex 3E (X1)
- Q9(d) - (f)
- Q10-18
- Q19, 20

2.3.4 The quadratic formula and the graph

Laws/Results

The quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

results in the roots of the equation $ax^2 + bx + c = 0$ being found.

Steps

Derivation of the quadratic formula to solve $ax^2 + bx + c = 0$

1. Rewrite as monic quadratic:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

2. Complete the square on x:

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

3. Simplify and rewrite with x as subject:

Further exercises

Ⓐ **Ex 3F**

- Q1-5, 9-10

ⓧ1 **Ex 3F**

- Q6-13
- Q14

2.3.5 The discriminant

Definition 18

The **discriminant** (Δ) of a quadratic equation:

$$\Delta = \ldots b^2 - 4ac \ldots$$

Laws/Results

Given the roots of a quadratic equation are located at: $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$

- $\Delta = 0$: ... one ... distinct (unique), ... real ... root. (Double root)

  Example

- $\Delta > 0$: ... two ... distinct (unique), ... real ... roots.
  - Rational roots when ... $\sqrt{\Delta}$ ... is rational, i.e. Δ is a ... perfect ... square ... .

  Example

- $\Delta < 0$: ... no ... real ... roots. (... two ... non ... real ... roots)

  Example

Example 34

Use the discriminant to determine the type of roots of the quadratic equation

(a) $y = 4x^2 - 4x + 1.$ (b) $y = 3x^2 - 5x + 5$

Example 35

For what values of λ does $x^2 - (\lambda + 5)x + 9 = 0$ have: **Answer:** (a) $1, -11$ (b) $\lambda \in (-11, 1)$

(a) equal roots (b) no real roots?

Example 36

Find the values of k for which $y = x^2 + (k - 1)x - (2k + 1)$ has unreal roots.

Answer: $k \in (-5, -1)$

Example 37

Show that the roots of the equation

$$4(m+1)x^2 - 4(m-1)x - 3 = 0 \qquad (m \neq -1)$$

are real, for all $m \in \mathbb{R}$.

Example 38

[Ex 8F Q22/1988 HSC] find, as a relation between k, ℓ and m, the condition for the quadratic equation in x

$$\left(k^2 + \ell^2\right) x^2 + 2\ell(k+m)x + \left(\ell^2 + m^2\right) = 0$$

to have real roots. Simplify your answer as far as possible.

Example 39

[2006 St George Girls' 2U]

(i) State the condition necessary for a quadratic equation to have rational roots. **1**

(ii) Prove that the equation $3px^2 = 2px + 3qx - 2q$, where p and q are rational, has rational roots for all values of p and q. **4**

(iii) What can be concluded about the number of roots if $p = \dfrac{3q}{2}$? **1**

Further exercises (Legacy Textbooks)

Ⓐ **Ex 10F**

- Q1-8, 11, 13-15

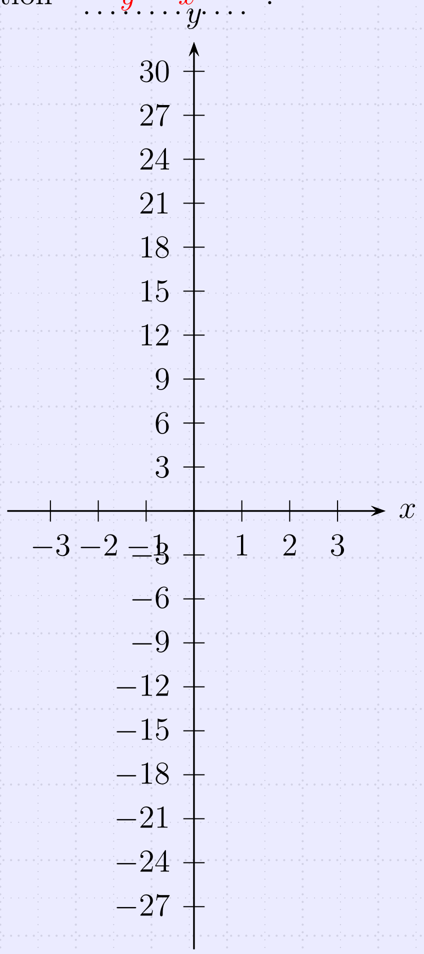
(X1) **Ex 8F**

- Q2-5 LC
- Q8-11, 21-22

2.4 Cubics (basic)

Definition 19

The basic cubic has equation $y = x^3$.



- Domain: $D = \{x : x \in \mathbb{R}\}$
- Range: $R = \{y : y \in \mathbb{R}\}$
- Generalised equation: $y = a(x-h)^3 + k$, a is the vertical stretch factor.
- The horizontal point of inflexion is at $(0, 0)$
- The cubic function in its most basic form is odd

Example 40

Sketch $y = x^3 - 3$, showing all important features.

Example 41

Sketch $y = (x-2)^3 + 3$, showing all important features.

2.5 Polynomials

2.5.1 Definitions

Definition 20

A *polynomial function* is a function with positive integer (or zero) powers of x.

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0$$

where $a_k \in \mathbb{R}$ are the *coefficients* of the corresponding power of x, and $n \in \mathbb{N}$.

- Domain: $D = \{x : x \in \mathbb{R}\}$
- Range: check graph!

Definition 21

The *leading term* is

$$a_n x^n$$

The *leading coefficient* is a_n.

Definition 22

A polynomial $P(x)$ is *monic* if $a_n = 1$.

Definition 23

The *constant term* is a_0, i.e. coefficient of x^0.

Definition 24

The *degree* of $P(x)$ is n, i.e. highest power of x.

Definition 25

Two polynomials $P(x)$ and $Q(x)$ are *identical* iff all their corresponding coefficients are equal, i.e.

$$\begin{aligned}\text{Given } P(x) &= a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 \\ Q(x) &= b_nx^n + b_{n-1}x^{n-1} + \cdots + b_2x^2 + b_1x + b_0\end{aligned}$$

then $P(x) \equiv Q(x)$ iff

$$\begin{aligned}a_n &= b_n \\ a_{n-1} &= b_{n-1} \\ &\vdots \\ a_1 &= b_1 \\ a_0 &= b_0\end{aligned}$$

Important note

Proficiency with various factorisation methods required

Example 42

Given $P(x) = x^3 - x^2 + x - 1$, $Q(x) = 3x^3 - 2x^2$ and $R(x) = -x^4 + 2x^3 - 3x^2$, evaluate

(a) $P(x)Q(x)$ (b) $Q(x)R(x)$

Answer: (a) $3x^6 - 5x^5 + 5x^4 - 5x^3 + 2x^2$ (b) $-3x^7 + 8x^6 - 13x^5 + 6x^4$

Further exercises

Ex 10A

- Q1-3 last column

2.5.2 Sketching simple polynomials

Definition 26

A fully factorised polynomial is of the form

$$y = a(x - x_1)(x - x_2)\cdots(x - x_k)$$

where x_1, x_2 etc are the ....zeros.... & a is the ....vertical.... ....dilation.... factor.

Theorem 2

Polynomial behaviour

Fully factored term	Algebra	Behaviour near zero	ⓧ1 Multiplicity
....Linear....			
....Quadratic....	$(x-2)^2$		
....Cubic....		Cubic	
....Quartic....			

Example 43

Sketch the following polynomials:

(a) $y = (x + 1)(x - 2)(x - 3)$

(b) $y = x(x - 2)^2$

(c) $y = (x - 2)^3(x + 1)$

(d) $y = -x(x - 1)(x - 2)(x + 1)$

Important note

All of the roots, plus the y intercept should be shown clearly.

Example 44

[2023 Adv HSC Q19]

(a) Sketch the graphs of the functions $f(x) = x - 1$ and $g(x) = (1 - x)(3 + x)$, showing the x intercepts. **2**

(b) Hence, or otherwise, solve the inequality $x - 1 < (1 - x)(3 + x)$ **2**

Exercises

1. Sketch the graphs for

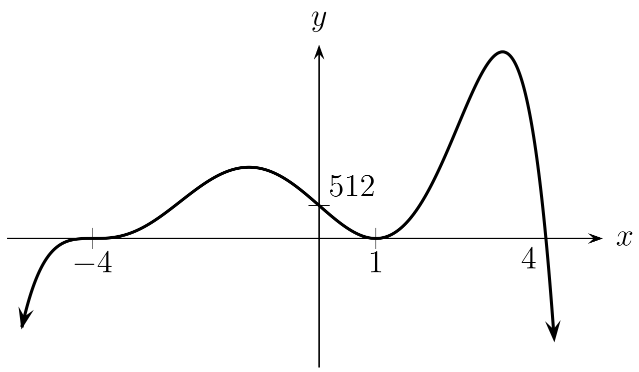
(a)	$y = (x-1)(x-3)$	(e)	$y = (x+1)^3$	(i)	$y = (x-1)^2(x-3)^2$
(b)	$y = x(x+2)(x-3)$	(f)	$y = x(x-3)^2$	(j)	$y = x^2(1-x)(x+1)$
(c)	$y = -x(x-1)(x-2)$	(g)	$y = (x+1)^2(x-1)$	(k)	$y = x^3(1-x)^3$
(d)	$y = (x-2)^2$	(h)	$y = -(x+2)(x-1)^3$	(l)	$y = 2(1-x)^4(x+2)^5$

2. Sketch the graphs for

(a) $y = x^3 + 3x^2 + 2x$ (b) $y = x^3 - 6x^2 + 9x$ (c) $y = x^3 + 2x^2 - x - 2$

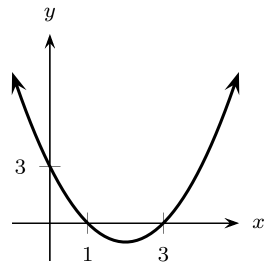
3. Sketch the following function: $f(x) = \begin{cases} -x^2(x+2) & x < 0 \\ x - x^3 & x \geq 0 \end{cases}$

4. Write down the equation of the lowest degree for the following polynomial graph:

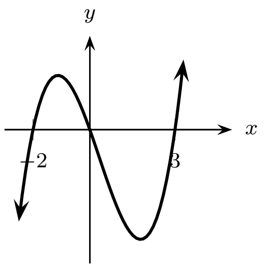


Answers

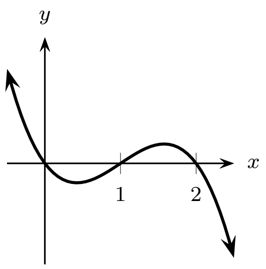
1. (a) $y = (x-1)(x-3)$



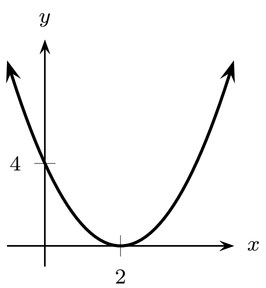
(b) $y = x(x+2)(x-3)$



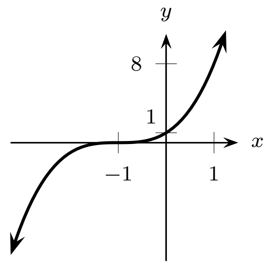
(c) $y = -x(x-1)(x-2)$



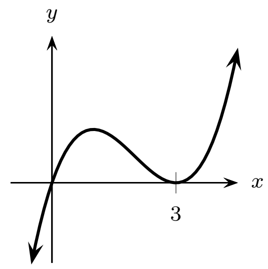
(d) $y = (x-2)^2$



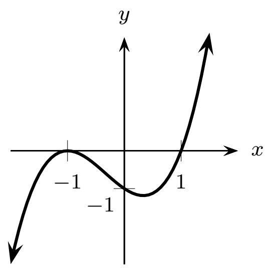
(e) $y = (x+1)^3$



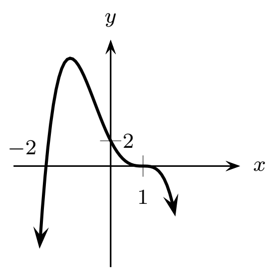
(f) $y = x(x-3)^2$



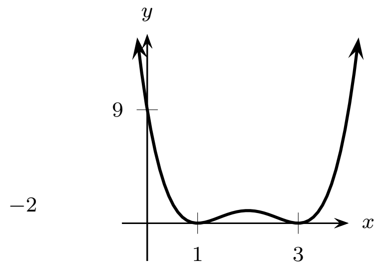
(g) $y = (x+1)^2(x-1)$



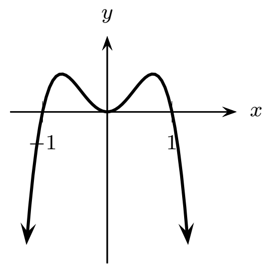
(h) $y = -(x+2)(x-1)^3$



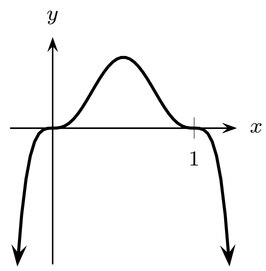
(i) $y = (x-1)^2(x-3)^2$



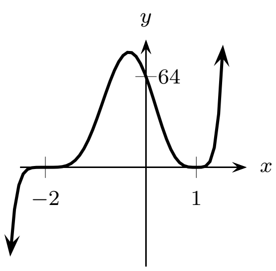
(j) $y = x^2(1-x)(x+1)$



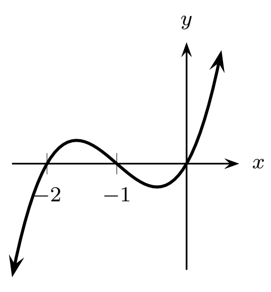
(k) $y = x^3(1-x)^3$



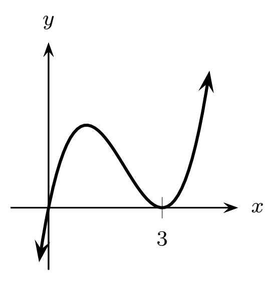
(l) $y = 2(1-x)^4(x+2)^5$



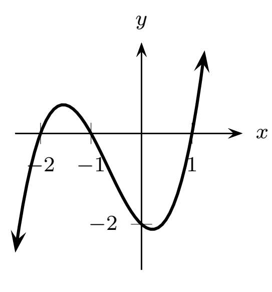
2. (a) $y = x^3 + 3x^2 + 2x$



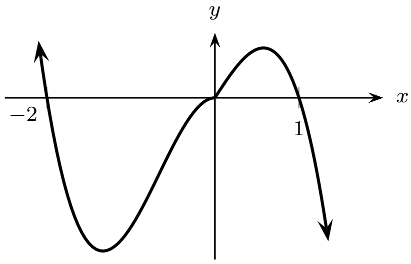
(b) $y = x^3 - 6x^2 + 9x$



(c) $y = x^3 + 2x^2 - x - 2$



3.



4. $y = -2(x-1)^2(x-4)(x+4)^3$

Further exercises (Legacy Textbooks)

Source McSeveney et al. (1986, Ex 15:07).

1. The following polynomials are given in factored form. Determine their zeros and sketch their graphs indicating clearly where they cut the x axis.

(a) $y = (x+1)(x-3)$

(b) $y = 2x(x-5)$

(c) $y = x(x+1)(x-1)$

(d) $y = (x-2)(x-1)(x+1)$

(e) $y = (x+4)(x+1)(x-2)$

(f) $y = x(x-3)(x+5)$

(g) $y = -x(x+2)(x-1)$

(h) $y = (3-x)(x+1)(x+2)$

(i) $y = (x-2)(x+1)(1-x)$

(j) $y = (2-x)(x+1)(x+4)$

(k) $y = x(x+2)(x+1)(x-3)$

(l) $y = (x+3)(x+1)(x-1)(3-x)$

2. Each polynomial function has a repeated root. Sketch the graph of each neatly.

(a) $y = (x-1)^2$

(b) $y = -(x+2)^2$

(c) $y = x(x+2)^2$

(d) $y = (x+1)(x-2)^2$

(e) $y = (x+3)^2(x-1)$

(f) $y = -x(x-1)^2$

(g) $y = x^2(5-x)$

(h) $y = (x+3)^3$

(i) $y = -(x+3)(x-5)^2$

(j) $y = (5-x)^3$

(k) $y = x(x+1)(x-2)^2$

(l) $y = (x+1)(x-2)^3$

GeoGebra

Check solutions with Geogebra

2.6 Circles

Definition 27

The general equation of a circle (relation) with radius r, centred at $C(h,k)$ is

$$(x-h)^2+(y-k)^2=r^2$$

- Domain: $D=\{x : x\in[h-r,h+r]\}$
- Range: $R=\{y : x\in[k-r,k+r]\}$

Derivation of general equation

Steps

1. Let

- $P(x,y)$ be the point that which is equidistant from a fixed point.
- (h,k) be the fixed point
- r be the fixed distance.

2. Apply distance formula:

....

....

....

....

Sketch

Definition 28

The general equation of a semicircle (function) with radius r, centred at $C(0,0)$ is

$$y = \sqrt{r^2 - x^2}$$

Sketch

Exercises

Source McSeveney et al. (1986, Ex 4:09)

1. Use the formula to find equation of the following circles:

(a) centre $(1,1)$ radius 7 units
(b) centre $(5,0)$ radius 2 units
(c) centre $(-3,-5)$ radius 4 units
(d) centre $(2,-5)$ radius 1 units
(e) centre $(0,2)$ radius $\frac{1}{2}$ units
(f) centre $(0,0)$ radius 3 units

2. Find the centre and radius of each of the following circles, then sketch each of the following on separate number planes:

(a) $(x-2)^2 + (y-3)^2 = 64$
(b) $(x+4)^2 + (y-1)^2 = 4$
(c) $(x+3)^2 + (y+3)^2 = 9$
(d) $(x-6)^2 + (y-5)^2 = 100$
(e) $x^2 + (y+5)^2 = 16$
(f) $(x-3)^2 + y^2 = 1$
(g) $x^2 + y^2 = 81$
(h) $x^2 + y^2 = 49$
(i) $x^2 + y^2 = 11$
(j) $(x-7)^2 + (y-8)^2 = 2$

3. Find the centre and radius of each of the following circles.

(a) $x^2 - 10x + y^2 + 8y + 32 = 0$
(b) $x^2 + y^2 + 8x - 14y = 35$
(c) $x^2 + y^2 - 18x - 20y + 60 = 0$
(d) $x^2 + y^2 - 9x + \frac{53}{4} = 0$

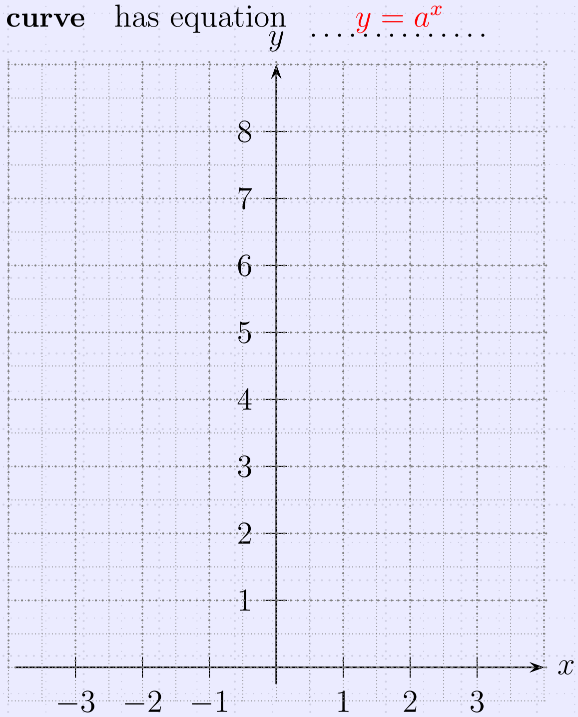
Answers

1. (a) $(x-1)^2+(y-1)^2 = 49$ (b) $(x-5)^2+(y-0)^2 = 4$ (c) $(x+3)^2+(y+5)^2 = 16$ (d) $(x-2)^2+(y+5)^2 = 1$ (e) $x^2+(y-2)^2 = \frac{1}{4}$ (f) $x^2 + (y-0)^2 = 9$ **2.** (a) centre $(2,3)$ radius 8 units (b) centre $(-4,1)$ radius 2 units (c) centre $(-3,-3)$ radius 3 units (d) centre $(6,5)$ radius 10 units (e) centre $(0,-5)$ radius 4 units (f) centre $(3,0)$ radius 1 units (g) centre $(0,0)$ radius 9 units (h) centre $(0,0)$ radius 7 units (i) centre $(0,0)$ radius $\sqrt{11}$ units (j) centre $(7,8)$ radius $\sqrt{2}$ units **3.** (a) centre $(5,-4)$ radius 3 units (b) centre $(-4,7)$ radius 10 units (c) centre $(9,10)$ radius 11 units (d) centre $(\frac{9}{2},0)$ radius $\sqrt{7}$ units

2.7 Exponential functions

Definition 29

A basic exponential curve has equation $y = a^x$



- Condition on a: $a > 0$
- Domain: $D = \{x : x \in \mathbb{R}\}$
- Range: $D = \{x : y \in (0, \infty)\}$
- When $x = 0$, $y = 1$
- When $x = 1$, $y = a$
- As
  - $x \to \infty$, $y \to \infty$

    $$\lim_{x\to\infty} a^x = \infty$$

  - $x \to -\infty$, $y \to 0^+$

    $$\lim_{x\to-\infty} a^x = 0^+$$

- Generalised equation: $y = ka^{x-b} + c$.

Important note

When sketching, always show two points on the curve!

Example 45

Sketch the graph of $y = \left(\frac{1}{2}\right)^x$, showing all important features.

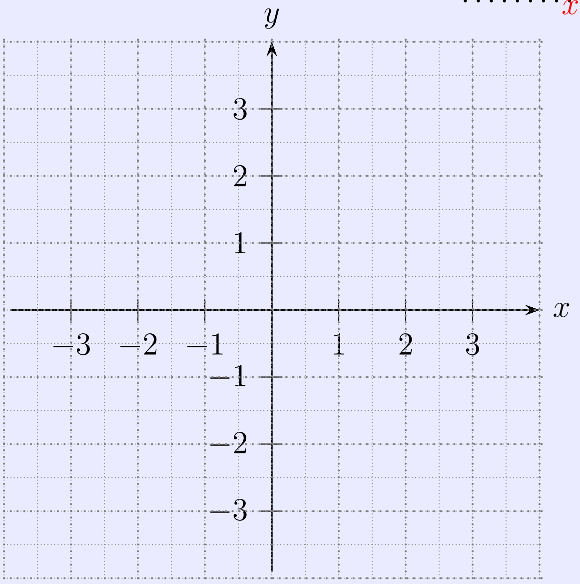
Example 46

Sketch the graph of $y = -2^{x+1}$, showing all important features.

2.8 Hyperbolas (upright rectangular)

Definition 30

The (upright rectangular) hyperbola has equation $y = \dfrac{1}{x}$.



- Domain: $D = \{x : x \in \mathbb{R} \setminus \{0\}\}$
- Range: $R = \{y : x \in \mathbb{R} \setminus \{0\}\}$
- Behaviour at extremities: as $x \to \pm\infty$, $y \to 0$, i.e.

$$\lim_{x \to \pm\infty} \frac{1}{x} = 0$$

- Behaviour near asymptotes: as $x \to 0^+$, $y \to \infty$, i.e.

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

  As $x \to 0^-$, $y \to -\infty$, i.e.

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

  This function is discontinuous at $x = 0$.

- Generalised equation: $y = \frac{a}{x-h} + k$

Important note

Other hyperbolae exist, but they may not be upright or rectangular. General equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where a is the length of the semi-major axis, and $a, b > 0$.

Example 47

Sketch $y = \dfrac{1}{x-2}$, showing all important features.

Example 48

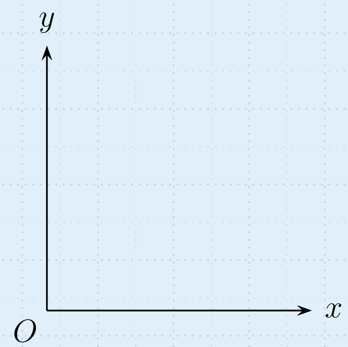
Sketch $y = \dfrac{-2}{x+1} + 3$, showing all important features.

Definition 31

If y is **inversely proportional** to x (or if y varies inversely with x), then .... $y = \dfrac{k}{x}$ ....

Fill in the spaces

- Graph of inverse variation $\left(y = \dfrac{k}{x}\right)$:



- If two variables are in inverse variation, then as one variable .... increases .... the other .... decreases ....

Example 49

[2023 CSSA Adv Trial Q13] (2 marks) The time it takes to complete a task varies inversely with the number of people assigned.

It takes 5 people to complete a task in 4 hours.

Find the amount of time it would take 8 people to complete the same task.

Answer: 2.5 hrs

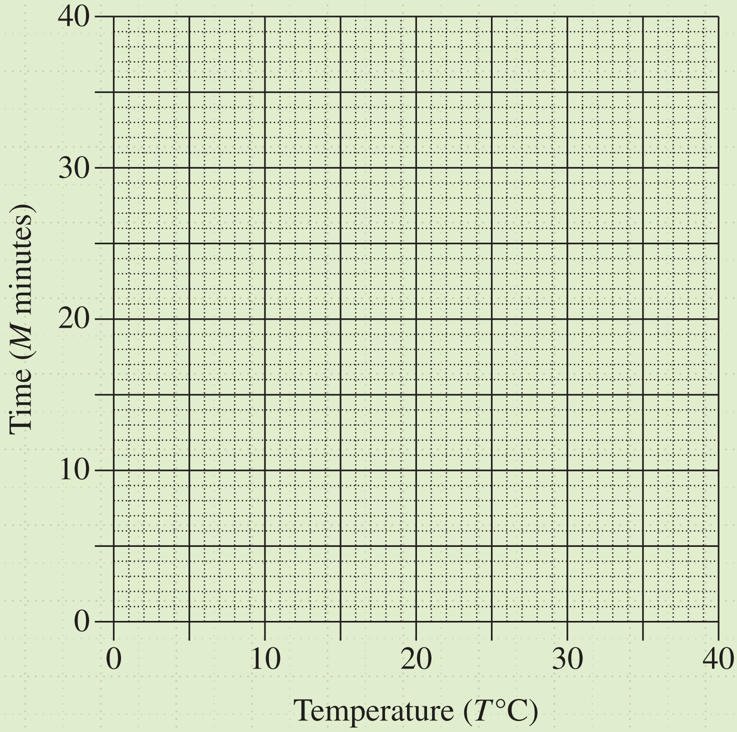
Example 50

[2022 Adv HSC Q12] A student believes that the time it takes for an ice cube to melt (M minutes) varies inversely with the room temperature (T°C). The student observes that at a room temperature of 15°C it takes 12 minutes for an ice cube to melt.

(a) Find the equation relating M and T. **2**

(b) By first completing this table of values, graph the relationship between temperature and time from $T = 5$°C to $T = 30$°C. **2**

T	5	15	30
M			



Further exercises

(A) **Ex 3G**

- Q12

(A) **Ex 3H**

- Q7-8, 10-12

(X1) **Ex 3G**

- Q13

(X1) **Ex 3H**

- Q5-7, 9-11, 13-17

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